The Hubbard model at half-filling, part III : the lower bound on the self-energy

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Abstract

We complete the proof that the two-dimensional Hubbard model at half-filling is *not* a Fermi liquid in the mathematically precise sense of Salmhofer, by establishing a lower bound on a second derivative in momentum of the first non-trivial self-energy graph.

I Introduction

This paper is the third of a series ([1]-[2]) devoted to the rigorous mathematical study of the two-dimensional Hubbard model at half-filling above the transition temperature to the expected low temperature region, which becomes Néel-ordered at zero temperature. The goal of this series was to prove that this model does not obey Salmhofer's criterion for Fermi liquid behavior of interacting Fermion systems at equilibrium ([3]-[4]). In this way, this model differs sharply from those with a Fermi surface close to the circle, which obey Salmhofer's criterion ([5]-[6]-[7]).

In the first paper [1] the convergent contributions of the model were bounded in the domain $|\lambda| \log^2 T \leq K$. In the second one [2], renormalization of the self-energy was performed to complete the proof of analyticity in the coupling constant of all the correlation functions in that domain. Salmhofer's criterion requires beyond this analyticity that the self-energy (in momentum space) is uniformly bounded together with its first and second derivatives in a domain $|\lambda||\log T| \leq K$. In this paper we prove that a certain second derivative of the self-energy at a particular value of the external momentum is *not* uniformly bounded in the domain $|\lambda| \log^2 T \leq K$ where we have established analyticity. This domain being smaller than the Salmhofer's one, it completes the proof that the two-dimensional half-filled Hubbard model is not a Fermi liquid. In conclusion, when we move from low filling to half-filling, the Hubbard model must undergo

a cross-over from Fermi to non-Fermi (in fact Luttinger) liquid behavior. This solves the controversy on the nature of two-dimensional Fermionic systems in their ordinary phase [8]. We refer to [1]-[2]-[4] for a more complete review and further references on mathematical study of interacting Fermions.

II Recall of notations

The two-dimensional Hubbard model is defined on the lattice $\mathbb{Z}^2$. Fixing a temperature $T > 0$, the "imaginary time", denoted x_0, belongs to the real interval $\left[-\frac{1}{T},\ \frac{1}{T}\right[$. In the following, we shall denote $\beta = \frac{1}{T}$. Indeed this interval should be thought of as a circle of length 2β, that is $\mathbb{R}/2\beta\mathbb{Z}$. Consequently, the momentum space, which is the dual of $\mathbb{R}/2\beta\mathbb{Z}\times\mathbb{Z}^2$ in the sense of the Fourier transform, is $\pi T\mathbb{Z}\times[\mathbb{R}/2\pi\mathbb{Z}]^2$. The torus $[\mathbb{R}/2\pi\mathbb{Z}]^2$ will be represented by the square $[-\pi,\ \pi[^2$, with periodic boundary conditions.

In Fourier variables, the expression of the propagator at half-filling reads :

$$C(k_0,k_1,k_2) = \frac{1}{ik_0 - \cos k_1 - \cos k_2} \tag{II.1}$$

if $k_0 = (2n+1)\pi T$ for some $n \in \mathbb{Z}$. If $k_0 = 2n\pi T$, $C(k_0,k_1,k_2) = 0$ because in the formalism of Fermionic theories at finite temperature, the propagator has an antiperiod β with respect to the x_0 variable and therefore each Fourier coefficient of even order vanishes. With a slight abuse of language, we can say that $C(k_0,k_1,k_2)$ is only defined for $k_0 = (2n+1)\pi T$. This set of values is called the Matsubara frequencies.

The expression of the propagator in real space is deduced by Fourier transform :

$$C(x_0,x_1,x_2) = \frac{1}{(2\pi)^3}\int dk_0 \int dk_1 \int dk_2\ \frac{e^{ik.x}}{ik_0 - \cos k_1 - \cos k_2} \tag{II.2}$$

where we adopt the notations of [1], namely the integral $\int dk_0$ really means the discrete sum over the Matsubara frequencies $2\pi T\sum_{n\in\mathbb{Z}}\eta((2n+1)\pi T)$ (with $k_0 = (2n+1)\pi T$), whereas the integrals over k_1 and k_2 are "true" integrals, for $(k_1,k_2)\in[-\pi,\ \pi[^2$. We have added an ultraviolet cutoff $\eta(k_0)$, which is a fixed C_0^∞ (which e.g. is 1 for $0 \le k_0 \le 1$ and 0 for $0 \le k_0 \ge 2$) in order to avoid some technicalities irrelevant for our main result, namely the fact that the integrand without this cutoff is not absolutely summable with respect to k_0 or n.

For our analysis, it will be convenient to introduce another parametrization of the spaces $[-\pi,\pi[^2$ and $\mathbb{Z}^2$. The idea is to "rotate" the Fermi surface of Figure II by an angle of $\frac{\pi}{4}$. In the $k_0 = 0$ plane, it is defined by $\cos k_1 + \cos k_2 = 0$, which is equivalent to $k_2 = \pi \pm k_1$ or $k_2 = -\pi \pm k_1$.

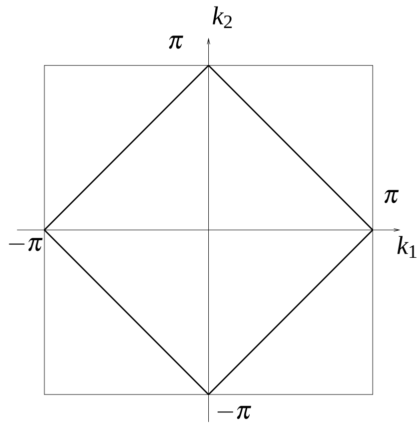

Figure 1: The square $[-\pi,\pi[^2$ and the Fermi surface

Introducing the variables $k_\pm = \frac{k_1 \pm k_2}{\pi} \iff \begin{cases} k_1 = \frac{\pi}{2}(k_+ + k_-) \\ k_2 = \frac{\pi}{2}(k_+ - k_-) \end{cases}$, the domain of integration $(k_1,k_2) \in [-\pi,\pi[^2$ becomes the set :

$$\mathscr{D} = \left\{ (k_+,k_-) \in [-2,2]^2 \text{ with } \begin{cases} -2 \le k_+ \le 0 \\ -2-k_+ \le k_- \le 2+k_+ \end{cases} \text{or } \begin{cases} 0 \le k_+ \le 2 \\ -2+k_+ \le k_- \le 2-k_+ \end{cases} \right\}. \tag{II.3}$$

As $\cos k_1 + \cos k_2 = 2\cos\frac{\pi}{2}k_+ \cos\frac{\pi}{2}k_-$, the Fermi surface in the variables $k_\pm$ is simply defined by $k_+ = \pm 1$, $k_- = \pm 1$. The new domain of integration, with the Fermi surface is represented on Figure 2.

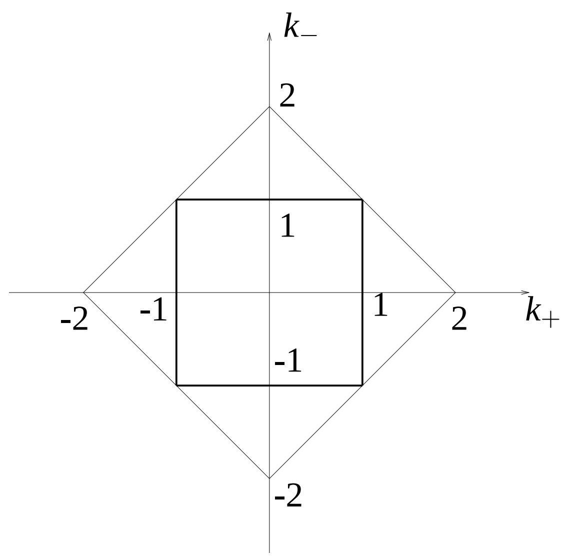

Figure 2: The domain of integration in (k_+,k_-) and the Fermi surface

In a dual way, we introduce new variables in real space, x_+ and x_- in such a way that $k_1x_1 +$

$k_2x_2 = k_+x_+ + k_-x_-$. We have :

$$\begin{cases} x_+ = \frac{\pi}{2}(x_1 + x_2) \\ x_- = \frac{\pi}{2}(x_1 - x_2)\ . \end{cases} \tag{II.4}$$

Observe that the image of the lattice $\mathbb{Z}^2$ by this change of variable is not $\frac{\pi}{2}\mathbb{Z}^2$ but the subset

$$S = \left\{ \left(\frac{\pi}{2}m, \frac{\pi}{2}n \right),\ (m,n) \in \mathbb{Z}^2,\ m \equiv n[2] \right\}\ . \tag{II.5}$$

In other words, the integers m and n must have same parity.

As the Jacobian of the transformation $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} & \frac{\pi}{2} \\ \frac{\pi}{2} & -\frac{\pi}{2} \end{pmatrix} \begin{pmatrix} k_+ \\ k_- \end{pmatrix}$ is $J = -\frac{\pi^2}{2}$, we have :

$$\int_{[-\pi,\pi]^2} dk_1 dk_2\ \frac{e^{i(k_1x_1+k_2x_2)}}{ik_0 - \cos k_1 - \cos k_2} = \frac{\pi^2}{2} \int_{\mathscr{D}} dk_+ dk_-\ \frac{e^{i(k_+x_++k_-x_-)}}{ik_0 - 2\cos\frac{\pi}{2}k_+ \cos\frac{\pi}{2}k_-}\ . \tag{II.6}$$

But the domain $\mathscr{D}$ is not very convenient for practical computations, and therefore we would like the k_+k_- integration domain to factorize. Since the complement set $[-2,2[^2 \backslash \mathscr{D}$ is another fundamental domain for the torus $\mathbb{R}^2/2\pi\mathbb{Z}^2$, we have :

$$\int_{\mathscr{D}} dk_+ dk_-\ \frac{e^{i(k_+x_++k_-x_-)}}{ik_0 - 2\cos\frac{\pi}{2}k_+ \cos\frac{\pi}{2}k_-} = \int_{[-2,2[^2\backslash\mathscr{D}} dk_+ dk_-\ \frac{e^{i(k_+x_++k_-x_-)}}{ik_0 - 2\cos\frac{\pi}{2}k_+ \cos\frac{\pi}{2}k_-}\ . \tag{II.7}$$

Hence :

$$\int_{\mathscr{D}} dk_+ dk_-\ \frac{e^{i(k_+x_++k_-x_-)}}{ik_0 - 2\cos\frac{\pi}{2}k_+ \cos\frac{\pi}{2}k_-} = \frac{1}{2} \int_{[-2,2]^2} dk_+ dk_-\ \frac{e^{i(k_+x_++k_-x_-)}}{ik_0 - 2\cos\frac{\pi}{2}k_+ \cos\frac{\pi}{2}k_-}\ . \tag{II.8}$$

Recapitulating, the expression of the propagator that we we take as our starting point is:

$$C(x_0, x_+, x_-) = \int d^3k \frac{e^{i(k_0x_0+k_+x_++k_-x_-)}}{ik_0 - 2\cos\frac{\pi}{2}k_+ \cos\frac{\pi}{2}k_-} \tag{II.9}$$

for $x_\pm$ satisfying the parity condition (II.5). In II.9 the notation $\int d^3k$ means

$$\frac{1}{32\pi} \int dk_0 \int_{[-2,2]^2} dk_+ dk_-\ , \tag{II.10}$$

where we recall that $\int dk_0$ means $2\pi T \sum_{n\in\mathbb{Z}} \eta((2n+1)\pi T))$, since $k_0 = (2n+1)\pi T$.

Now, let us consider, in Fourier space, the amplitude of the graph G represented on Figure 3, with an incoming momentum $k = (k_0, k_+, k_-)$. This amplitude is denoted $A_G(k)$ and written explicitly as $A_G(k_0, k_+, k_-) = \int d^3x\, C(x)\bar{C}(x)^2 e^{-ik.x}$ (where arrows join antifields to fields).

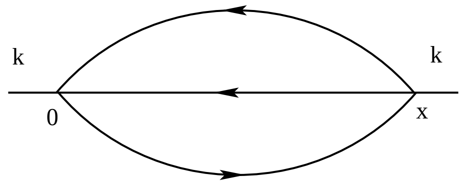

Figure 3: The first non-trivial graph contributing to the self-energy

More precisely, we shall consider the second momentum derivative in the $+$ direction of this quantity :

$$\partial_+^2 A_G(k) = \int d^3x \, x_+^2 C(x) \overline{C(x)}^2 e^{-ik.x} \; . \tag{II.11}$$

The quantity we are going to study is explicitly written :

$$\partial_+^2 A_G(\pi T,1,0) = \int d^3x \, x_+^2 \int d^3k_1 \frac{e^{ik_1.x}}{ik_{1,0} - 2\cos\frac{\pi}{2}k_{1,+}\cos\frac{\pi}{2}k_{1,-}} \int d^3k_2 \frac{e^{ik_2.x}}{-ik_{2,0} - 2\cos\frac{\pi}{2}k_{2,+}\cos\frac{\pi}{2}k_{2,-}} \int d^3k_3 \frac{e^{ik_3.x}}{-ik_{3,0} - 2\cos\frac{\pi}{2}k_{3,+}\cos\frac{\pi}{2}k_{3,-}} \, e^{i(\pi T x_0 + x_+)} \; , \tag{II.12}$$

where again $\int d^3x$ includes the parity condition (II.5). We state now the main result of this paper :

Theorem II.1 *There exists some strictly positive constant K such that, for T small enough :*

$$\left|\partial_+^2 A_G(\pi T,1,0)\right| \geq \frac{K}{T} \; . \tag{II.13}$$

We recall that this result, joined to the analysis of [2], leads to the result that the self-energy of the model is not uniformly $\mathscr{C}^2$ in the domain $|\lambda|\log^2 T < K$ and therefore that the two-dimensional Hubbard model at half-filling is not a Fermi liquid.

III Plan of the proof

Theorem (II.1) will be proven thanks to a sequence of lemmas. But before presenting these lemmas, let us give an overview of our strategy. We use the sector decomposition introduced in [1] to write :

$$\partial_+^2 A_G(\pi T,1,0) = \sum_{\sigma_1,\sigma_2,\sigma_3} \int d^3x \, x_+^2 C_{\sigma_1}(x) \bar{C}_{\sigma_2}(x) \bar{C}_{\sigma_3}(x) e^{-i(\pi T x_0 + x_+)} \; , \tag{III.14}$$

where a sector σ is a triplet (i, s_+, s_-) with $0 \leq s_\pm \leq i$ and $s_+ + s_- \leq i$.

The main idea is that in the sum over sectors of equation (III.14), the leading contribution is given by a restricted sum corresponding to sectors close to the ”vertical part” of the Fermi

surface, defined by $k_+ = \pm 1$. To express this more precisely, let Λ be an integer (whose value will be chosen later), which will play the role of a cut-off for the sectors. We want to prove that as soon as one sector is not close to $k_+ = \pm 1$, then we have a small contribution. Let us denote $\sum^{\Lambda}_{\{i_j\},\{s_j^+\},\{s_j^-\}}$ the sum in which at least one sector is "far" from the vertical sides of the Fermi surface. Precisely, this means that at least one index s_j^+ is smaller than $i_{max}(T) - \Lambda$, where, as in [1], $M^{-i_{max}(T)} \approx T$. This constrained sum can be written explicitly :

$$\sum^{\Lambda}_{\{i_j\},\{s_j^+\},\{s_j^-\}} = \sum_{i_1,s_1^-,\sigma_2,\sigma_3} \sum_{s_1^+=0}^{\inf(i_1,i_{max}(T)-\Lambda} + \sum_{i_1,s_1^-,i_2,s_2^-,\sigma_3} \sum_{s_1^+=i_{max}(T)-\Lambda}^{i_1} \sum_{s_2^+=0}^{\inf(i_2,i_{max}(T)-\Lambda} + \sum_{i_1,s_1^-,i_2,s_2^-,i_3,s_3^-} \sum_{s_1^+=i_{max}(T)-\Lambda}^{i_1} \sum_{s_2^+=i_{max}(T)-\Lambda}^{i_2} \sum_{s_3^+=0}^{\inf(i_3,i_{max}(T)-\Lambda} . \quad \text{(III.15)}$$

Defining :

$$A_G^{\Lambda}(\pi T,1,0) = \sum^{\Lambda}_{\{i_j\},\{s_j^+\},\{s_j^-\}} \int d^3x\, C_{\sigma_1}(x)\bar{C}_{\sigma_2}(x)\bar{C}_{\sigma_3}(x)e^{-i(\pi T x_0 + x_+)} , \quad \text{(III.16)}$$

we write :

$$\partial_+^2 A_G(\pi T,1,0) = \partial_+^2 A_{G,\Lambda}(\pi T,1,0) + \partial_+^2 A_G^{\Lambda}(\pi T,1,0) \quad \text{(III.17)}$$

where $\partial_+^2 A_{G,\Lambda}(\pi T,1,0) = \partial_+^2 A_G(\pi T,1,0) - \partial_+^2 A_G^{\Lambda}(\pi T,1,0)$ is expressed as a sum over sectors that are all close to $k_+ = \pm 1$, i.e. such that each s_j^+ index is greater than $i_{max}(T) - \Lambda$.

Each sector appearing in the sum expressing $\partial_+^2 A_{G,\Lambda}(\pi T,1,0)$ will be divided into two disjoint subsectors, according to the sign of $\cos\frac{\pi}{2}k_+$. We recall that in [1], the sectors were defined as :

$$\left| ik_0 - 2\cos\frac{\pi}{2}k_+\cos\frac{\pi}{2}k_- \right| \approx M^{-i} , \quad \left|\cos\frac{\pi}{2}k_+\right| \approx M^{-s^+} , \quad \left|\cos\frac{\pi}{2}k_-\right| \approx M^{-s^-} . \quad \text{(III.18)}$$

We shall call σ^r and σ^l ("right" and "left") the subdomains of σ corresponding to $\cos\frac{\pi}{2}k_+ < 0$ and $\cos\frac{\pi}{2}k_+ > 0$ respectively. The underlying motivation is that, if a momentum, say k_1, is close to the side $k_+ = 1$, by momentum conservation at each vertex, the other ones are necessarily close to the other side $k_+ = -1$. Let us state precisely this point :

Lemma III.1 *In the sum expressing $\partial_+^2 A_{G,\Lambda}(\pi T,1,0)$, if one sector is of the right type, then the other ones are of the left type.*

The proof is obvious by momentum conservation in the $+$ direction. We conclude that :

$$\partial_+^2 A_{G,\Lambda}(\pi T,1,0) = \sum_{\substack{\{\sigma_j\},i_j,s_j^+>i_{max}(T)-\Lambda \\ \sigma_1 \text{ right}}} \int d^3x\, x_+^2 C_{\sigma_1}(x)\bar{C}_{\sigma_2}(x)\bar{C}_{\sigma_3}(x)e^{-i(\pi T x_0+x_+)}$$
$$+ \sum_{\substack{\{\sigma_j\},i_j,s_j^+>i_{max}(T)-\Lambda \\ \sigma_2 \text{ right}}} \int d^3x\, x_+^2 C_{\sigma_1}(x)\bar{C}_{\sigma_2}(x)\bar{C}_{\sigma_3}(x)e^{-i(\pi T x_0+x_+)}$$
$$+ \sum_{\substack{\{\sigma_j\},i_j,s_j^+>i_{max}(T)-\Lambda \\ \sigma_3 \text{ right}}} \int d^3x\, x_+^2 C_{\sigma_1}(x)\bar{C}_{\sigma_2}(x)\bar{C}_{\sigma_3}(x)e^{-i(\pi T x_0+x_+)} \ . \quad \text{(III.19)}$$

Among these three contributions, the last two ones are indeed equal, and we have :

$$\partial_+^2 A_{G,\Lambda}(\pi T,1,0) = \sum_{\substack{\{\sigma_j\},i_j,s_j^+>i_{max}(T)-\Lambda \\ \sigma_1 \text{ right}}} \int d^3x\, x_+^2 C_{\sigma_1}(x)\bar{C}_{\sigma_2}(x)\bar{C}_{\sigma_3}(x)e^{-i(\pi T x_0+x_+)}$$
$$+2 \sum_{\substack{\{\sigma_j\},i_j,s_j^+>i_{max}(T)-\Lambda \\ \sigma_2 \text{ right}}} \int d^3x\, x_+^2 C_{\sigma_1}(x)\bar{C}_{\sigma_2}(x)\bar{C}_{\sigma_3}(x)e^{-i(\pi T x_0+x_+)} \ . \quad \text{(III.20)}$$

In each sum, we replace the $\cos\frac{\pi}{2}k_+$ appearing in the propagators by their Taylor expansions in the neighborhood of $+1$ in a right sector, and in a neighborhood of -1 in a left sector. We have $\cos\frac{\pi}{2}k_+ \approx -\frac{\pi}{2}(k_+-1)$ for k_+ in the neighborhood of 1, in which case we put $q_+ = (k_+-1)$ and $\cos\frac{\pi}{2}k_+ \approx \frac{\pi}{2}(k_++1)$ for k_+ in the neighborhood of -1, in which case we put $q_+ = (k_++1)$. This replacement gives an expression that we call $\partial_+^2 \tilde{A}_{G,\Lambda}(\pi T,1,0)$:

$$\partial_+^2 \tilde{A}_{G,\Lambda}(\pi T,1,0) = \int d^3x\, x_+^2 \int d^3k_1\, \frac{u_\Lambda(q_{1,+})e^{ik_1.x}}{ik_{1,0}+\pi q_{1,+}\cos\frac{\pi}{2}k_{1,-}}$$
$$\int d^3k_2\, \frac{u_\Lambda(q_{2,+})e^{ik_2.x}}{-ik_{2,0}-\pi q_{2,+}\cos\frac{\pi}{2}k_{2,-}} \int d^3k_3\, \frac{u_\Lambda(q_{3,+})e^{ik_3.x}}{-ik_{3,0}-\pi q_{3,+}\cos\frac{\pi}{2}k_{3,-}} e^{-i(\pi T x_0+x_+)}$$
$$+2\int d^3x\, x_+^2 \int d^3k_1\, \frac{u_\Lambda(q_{1,+})e^{ik_1.x}}{ik_{1,0}-\pi q_{1,+}\cos\frac{\pi}{2}k_{1,-}}$$
$$\int d^3k_2\, \frac{u_\Lambda(q_{2,+})e^{ik_2.x}}{-ik_{2,0}+\pi q_{2,+}\cos\frac{\pi}{2}k_{2,-}} \int d^3k_3\, \frac{u_\Lambda(q_{3,+})e^{ik_3.x}}{-ik_{3,0}-\pi q_{3,+}\cos\frac{\pi}{2}k_{3,-}} e^{-i(\pi T x_0+x_+)} \ , \quad \text{(III.21)}$$

where the $u_\Lambda(q_+)$'s is now the smooth scaled cutoff function $u(M^{i_{max}(T)-\Lambda}q_+)$ which expresses the former sector constraint $s_+ \geq i_{max}(T)-\Lambda$ (recall that u is our fixed basic cutoff function).

In (III.21) we can freely change each integral over dk_+ which ran over $[-2,2]$ into an integral on dq_+ which runs from $[-\infty,\infty]$. We still denote $\int d^3k$ the corresponding integrals.

We write now for each propagator in (III.21), $u_\Lambda(q_+) = 1+u^1(q_+)+u_1^\Lambda(q_+)$ where $u^1(q_+) = u(q_+)-1$ and $u_1^\Lambda(q_+) = u_\Lambda(q_+)-u(q_+)$. In this way we generate three terms:

- one in which all three functions $u_\Lambda(q_+)$ are replaced by 1. We call this term $\partial_+^2 \tilde{A}_G(\pi T,1,0)$
- one in which there is at least one factor $u_1^\Lambda(q_+)$ and no factor $u^1(q_+)$. We call this term $\partial_+^2 A_{G,1}^\Lambda(\pi T,1,0)$.
- finally one in which there is at least one factor $u^1(q_+)$. We call this term $\partial_+^2 A_G^1(\pi T,1,0)$.

At this stage, we recapitulate :

$$\begin{aligned} \partial_+^2 A_G(\pi T,1,0) &= \partial_+^2 \tilde{A}_G(\pi T,1,0) + \partial_+^2 A_{G,1}^\Lambda(\pi T,1,0) + \partial_+^2 A_G^1(\pi T,1,0) \\ &+ \Big(\partial_+^2 A_{G,\Lambda}(\pi T,1,0) - \partial_+^2 \tilde{A}_{G,\Lambda}(\pi T,1,0)\Big) + \partial_+^2 A_G^\Lambda(\pi T,1,0)\ . \end{aligned} \tag{III.22}$$

This relation shows that the quantity under study, $\partial_+^2 A_G(\pi T,1,0)$, is equal to the approximation $\partial_+^2 \tilde{A}_G(\pi T,1,0)$, up to the four error terms

$$\partial_+^2 A_G^\Lambda(\pi T,1,0),\ \partial_+^2 A_{G,1}^\Lambda(\pi T,1,0),\ \Big(\partial_+^2 A_{G,\Lambda}(\pi T,1,0) - \partial_+^2 \tilde{A}_{G,\Lambda}(\pi T,1,0)\Big),\ \partial_+^2 A_G^1(\pi T,1,0)\ . \tag{III.23}$$

Now we are going to prove a lower bound similar to the one of Theorem II.1, but on the quantity $\partial_+^2 \tilde{A}_G(\pi T,1,0)$, and establish an upper bound on each of the four error terms. More precisely, if we have $\left|\partial_+^2 \tilde{A}_G(\pi T,1,0)\right| > \frac{K}{T}$ for some constant $K>0$ and if the modulus of each error term is smaller than $\frac{K'}{T}$ with $K' << K$, we shall conclude that :

$$\left|\partial_+^2 A_G(\pi T,1,0)\right| > \frac{K-4K'}{T}\ , \tag{III.24}$$

which shall prove Theorem II.1. The result that $\left|\partial_+^2 \tilde{A}_G(\pi T,1,0)\right| > \frac{K}{T}$ is really the most difficult to establish, and its proof is the heart of this paper. But the control of the error terms is easier, and each one will correspond to a lemma. We shall begin by these lemmas in next section, and then turn to the lower bound on $\left|\partial_+^2 \tilde{A}_G(\pi T,1,0)\right|$.

IV The control of the error terms

First we state a result that is not necessary for proving Theorem II.1 but whose proof illustrates the way the sector decomposition allows us to establish quite easily upper bounds.

Lemma IV.1 *There exists some constant $K_1>0$ such that :*

$$\left|\partial_+^2 A_G(\pi T,1,0)\right| \le \frac{K_1}{T}\ . \tag{IV.25}$$

Proof : We use the decay property of $C_{(i,s_+,s_-)}(x)$ (see [1], Lemma 1) :

$$\left|C_{(i,s_+,s_-)}(x)\right| \leq c.M^{-s_+-s_-}\exp\left(-c\Big(d_\sigma(x)\Big)^\alpha\right) \ , \tag{IV.26}$$

where $\alpha \in]0,1[$ is a fixed number, c is a constant and $d_\sigma(x) = M^{-i}|x_0| + M^{-s_+}|x_+| + M^{-s_-}|x_-|$. We have :

$$\begin{aligned}\left|\partial_+^2 A_G(k)\right| \leq c^3.&M^{-\sum_{j=1}^3 s_j^+ - \sum_{j=1}^3 s_j^-} \\ &\sum_{\{i_j\},\{s_j^+\},\{s_j^-\}} \int d^3x\, x_+^2 \exp\left(-c\sum_{j=1}^3\left(M^{-i_j}|x_0| + M^{s_j^+}|x_+| + M^{-s_-}|x_-|\right)^\alpha\right) \ .\end{aligned} \tag{IV.27}$$

Among the indices i_1, i_2 and i_3, we keep the best one, i. e. the smallest one, to perform the integration over x_0. We proceed in an analogous way for the indices (s_1^+, s_2^+, s_3^+) and (s_1^-, s_2^-, s_3^-) respectively. Thus we have :

$$\left|\partial_+^2 A_G(k)\right| \leq c^3 \sum_{\{i_j\},\{s_j^+\},\{s_j^-\}} M^{-\sum_{j=1}^3 s_j^+ - \sum_{j=1}^3 s_j^-} M^{\inf\{i_j\}} M^{3\inf\{s_j^+\}} M^{\inf\{s_j^-\}} \ . \tag{IV.28}$$

To carry out our discussion, we introduce several notations. If (a_1, a_2, a_3) is a family of three (not necessarily distinct) real numbers, we denote as usual $\inf\{a_j\}$ the smallest number among the a_j's, but we define also

$$\inf_2\{a_j\} = \inf\Big(\{a_1,a_2,a_3\}\backslash\{\inf\{a_1,a_2,a_3\}\}\Big) \tag{IV.29}$$

and :

$$\inf_3\{a_j\} = \inf\Big(\{a_1,a_2,a_3\}\backslash\{\inf\{a_1,a_2,a_3\},\inf_2\{a_1,a_2,a_3\}\}\Big) \ . \tag{IV.30}$$

Remark that $\inf_3\{a_j\}$ is indeed $\sup\{a_j\}$. Finally in this paragraph we shall write simply $\sum a_j$ instead of $\sum_{j=1}^3 a_j$, and similarly for the s_j^+'s and the s_j^-'s. With these notations, it is very easy to check the following identity :

$$\inf\{a_j\} = \frac{1}{3}\sum_{j=1}^3 a_j - \frac{1}{3}\Big[\inf_2\{a_j\} - \inf\{a_j\}\Big] - \frac{1}{3}\Big[\inf_3\{a_j\} - \inf\{a_j\}\Big] \ . \tag{IV.31}$$

We introduce the abbreviation :

$$\Delta\{a_j\} = \Big[\inf_2\{a_j\} - \inf\{a_j\}\Big] + \Big[\inf_3\{a_j\} - \inf\{a_j\}\Big] \ , \tag{IV.32}$$

so that we have :

$$\inf\{a_j\} = \frac{1}{3}\sum a_j - \frac{1}{3}\Delta\{a_j\} \ . \tag{IV.33}$$

We use this identity to replace $\inf\{i_j\}$ and $\inf\{s_j^\pm\}$ in formula (IV.28), and we obtain :

$$\left|\partial_+^2 A_G(k)\right| \le c^3 \sum_{\{i_j\},\{s_j^+\},\{s_j^-\}} M^{-\sum s_j^+ - \sum s_j^-} M^{\frac{1}{3}\sum i_j - \frac{1}{3}\Delta\{i_j\}} M^{\sum s_j^+ - \Delta\{s_j^+\}} M^{\inf\{s_j^-\}} \ . \tag{IV.34}$$

Since $\inf\{s_j^-\} \le \frac{1}{3}\sum s_j^-$, we can write :

$$\left|\partial_+^2 A_G(k)\right| \le c^3 \sum_{\{i_j\},\{s_j^+\};\{s_j^-\}} M^{\frac{1}{3}\sum i_j - \frac{1}{3}\Delta\{i_j\}} M^{-\Delta\{s_j^+\}} M^{-\frac{2}{3}\sum s_j^-} \ . \tag{IV.35}$$

Now, we use the constraints in the sum $\sum_{\{i_j\},\{s_j^+\};\{s_j^-\}}$ to write, for each $j \in \{1,2,3\}$:

$$s_j^- \ge i_j - s_j^+ - 2 \ . \tag{IV.36}$$

We deduce that :

$$\frac{1}{3}\sum s_j^- \ge \frac{1}{3}\sum i_j - \frac{1}{3}\sum s_j^+ - 2 \tag{IV.37}$$

and

$$M^{-\frac{1}{3}\sum s_j^-} \le M^2 M^{-\frac{1}{3}\sum i_j + \frac{1}{3}\sum s_j^+} \ . \tag{IV.38}$$

Replacing in equation (IV.35), we get :

$$\left|\partial_+^2 A_G(k)\right| \le c^3 M^2 \sum_{\{i_j\},\{s_j^+\},\{s_j^-\}} M^{-\frac{1}{3}\Delta\{i_j\}} M^{\frac{1}{3}\sum s_j^+ - \Delta\{s_j^+\}} M^{-\frac{1}{3}\sum s_j^-} \ , \tag{IV.39}$$

and using relation (IV.33), we have :

$$\left|\partial_+^2 A_G(k)\right| \le c^3 M^2 \sum_{\{i_j\},\{s_j^+\},\{s_j^-\}} M^{-\frac{1}{3}\Delta\{i_j\}} M^{\inf\{s_j^+\} - \frac{2}{3}\Delta\{s_j^+\}} M^{-\frac{1}{3}\sum s_j^-} \ . \tag{IV.40}$$

At last, let us denote κ the value of the index j such that $s_\kappa^+ = \inf\{s_j^+\}$. We write $\inf\{s_j^+\} = i_\kappa - (i_\kappa - s_\kappa^+)$. Finally we obtain :

$$\left|\partial_+^2 A_G(k)\right| \le c^3 M^2 \sum_{\{i_j\},\{s_j^+\},\{s_j^-\}} M^{i_\kappa} M^{-\frac{1}{3}\Delta\{i_j\}} M^{-(i_\kappa - s_\kappa^+) - \frac{2}{3}\Delta\{s_j^+\}} M^{-\frac{1}{3}\sum s_j^-} \ . \tag{IV.41}$$

Clearly the sums over s_1^-, s_2^- and s_3^- can be bounded by $K_2 = \frac{M}{(M^{1/3}-1)^3}$. The decay $M^{-\frac{2}{3}\Delta\{s_j^+\}}$ can be used to perform the sums over s_j^+ for $j \ne \kappa$, also at a cost K_2. In the same way, we use the decay $M^{-\frac{1}{3}\Delta\{i_j\}}$ to sum over the values i_j, $j \ne \kappa$ also at cost K_2 per sum. It remains to sum over s_κ^+ :

$$\sum_{0 \le s_\kappa^+ \le i_\kappa} M^{-(i_\kappa - s_\kappa^+)} \le \frac{M}{M-1} \ . \tag{IV.42}$$

At last, we have :

$$\left|\partial_+^2 A_G(k)\right| \leq K \sum_{i_\kappa=0}^{i_{max}(T)} M^{i_\kappa} = K\frac{M^{i_{max}(T)+1}}{M-1} \tag{IV.43}$$

and we have $M^{i_{max}(T)} \sim \frac{1}{T}$ (see [1]), which proves lemma IV.1. ∎

We have then the following lemma, which is a slight refinement of lemma IV.1 :

Lemma IV.2

$$\left|\partial_+^2 A_G^\Lambda(\pi T,1,0)\right|, \left|\partial_+^2 A_{G,1}^\Lambda(\pi T,1,0)\right| \leq \frac{K_1}{M^\Lambda T} \tag{IV.44}$$

where K_1 is the constant of Lemma IV.1.

Proof : It is similar to the proof of Lemma IV.1. The case of $\partial_+^2 A_{G,1}^\Lambda(\pi T,1,0)$ can be decomposed into sectors exactly in the same way than $\partial_+^2 A_G^\Lambda(\pi T,1,0)$ because away from the singularity and in a bounded domain in k_+, the presence of πq_+ instead of $\cos\frac{\pi}{2}k_+$ does not change anything to the bounds on the propagators in sectors. Each step is then similar to the proof of of lemma IV.1 until we arrive at the last sum, for which :

$$\begin{aligned} \sum_{i_\kappa=0}^{i_{max}(T)-\Lambda} M^{i_\kappa} &= \frac{M^{i_{max}(T)-\Lambda+1}-1}{M-1} \qquad \text{(IV.45)}\\ &\leq \frac{M}{M-1}.\frac{M^{i_{max}(T)}}{M^\Lambda} = \frac{K'}{T.M^\Lambda}\ , \qquad \text{(IV.46)} \end{aligned}$$

which proves the lemma. ∎

The following lemma bounds the contributions with at least one large infrared cutoff u^1 on one propagator :

Lemma IV.3

$$\left|\partial_+^2 A_G^1(\pi T,1,0)\right| \leq K_2 \tag{IV.47}$$

where K_2 is some new constant.

Proof : The main idea is that a propagator bearing cutoff $u^1 = 1-u$ on q_+ decays on a length scale $O(1)$ in x_+, so the factor x_+^2 in $\partial_+^2 A_G^1$ is now harmless, and this prevents the divergence in $1/T$ of the bound.

We remark first that in the amplitude $\partial_+^2 A_G^1$ we can change the sum over x_+ into a sum over the non zero values of x_+, because of the x_+^2 integrand. Since a propagator bearing cutoff $u^1 = 1-u$ on q_+ is not absolutely integrable at large q_+, we first prepare all such propagators (there are between 1 and 3 of them) using integration by parts.

For any such propagator we first split the q_+ integration into the two regions $\int_1^\infty dq_+$ and $\int_{-\infty}^{-1} dq_+$ and treat only the first term, the other one being identical. Similarly we can assume

that we work on a 'right' propagator, so that $q_+ = k_+ - 1$, the other case being identical. The corresponding object is then:

$$
\begin{aligned}
D(x) &= e^{ix_+} \int dk_0 \int_{-2}^{2} dk_- \int_1^\infty dq_+ \, \frac{[1-u(q_+)] e^{i(k_0x_0+k_-x_-+q_+x_+)}}{ik_0 + \pi q_+ \cos\frac{\pi}{2}k_-} \\
&= -\frac{ie^{ix_+}}{x_+} \int dk_0 \int_{-2}^{2} dk_- \int_1^\infty dq_+ \Bigg[\frac{[\pi \cos\frac{\pi}{2}k_-][1-u(q_+)] e^{i(k_0x_0+k_-x_-+q_+x_+)}}{[ik_0 + \pi q_+ \cos\frac{\pi}{2}k_-]^2} \\
&\qquad + \frac{u'(q_+) e^{i(k_0x_0+k_-x_-+q_+x_+)}}{ik_0 + \pi q_+ \cos\frac{\pi}{2}k_-} \Bigg] . \qquad \text{(IV.48)}
\end{aligned}
$$

The last term, having a compact support u' is similar to the ones of the previous lemma, and left to the reader. Let us treat the first term.

We divide it with a partition of unity into new sectors i, s_+, s_- according to the size of the denominator $ik_0 + \pi q_+ \cos\frac{\pi}{2}k_-$, which is M^{-i}, the size of q_+ which is now of order M^{+s_+}, with $s_+ > 0$, and of k_- which is of order $M^{-s_-} = M^{-i-s_+}$, with $s_- = i + s_+$. The bounds are:

$$
\begin{aligned}
|D_{i,s_+,s_-}(x)| &\leq K|x_+|^{-1} M^{+i} M^{s_+} M^{-2s_-} e^{-c[M^{-i}x_0 + M^{s_+}x_+ + M^{-s_-}x_-]^\alpha} \\
&\leq \frac{2K}{\pi} M^{-i-s_+} e^{-c[M^{-i}x_0 + M^{s_+}x_+ + M^{-s_-}x_-]^\alpha} , \qquad \text{(IV.49)}
\end{aligned}
$$

since for non zero x_+, on the tilted lattice $|x_+|^{-1}$ is bounded by $2/\pi$. Hence taking into account that the "integral" $\int dx_+$ is really a discrete sum on $\frac{\pi}{2}\mathbb{Z}$:

$$
\int dx_+ x_+^2 |D_{i,s_+,s_-}(x)| \leq K M^{-i-3s_+} e^{-[M^{-i}x_0 + M^{s_+}x_+ + M^{-s_-}x_-]^\alpha/2} . \qquad \text{(IV.50)}
$$

Finally we need to optimize the dx_0 and dx_- using the best of the three other propagators. This leads to a bound which obviously is uniform in T. For instance if the three propagators have large infrared cutoffs $u^1 = 1 - u$, we get the bound

$$
\sum_{\substack{i_1,i_2,i_3 \\ s_{+,1},s_{+,2},s_{+,3}}} K M^{-\sum_j i_j - \sum_j s_{+,j} - 2\sup s_{+,j} + \inf\{i\} + \inf\{i+s_+\}} \leq \sum_{\substack{i_1,i_2,i_3 \\ s_{+,1},s_{+,2},s_{+,3}}} K M^{-(1/3)\sum_j i_j - (4/3)\sum_j s_{+,j}} \leq K' , \qquad \text{(IV.51)}
$$

and the other cases, when one or two propagators are of ordinary type, are similar and left to the reader. ∎

Finally we state the lemma that allows us to control the replacement of $\cos\frac{\pi}{2}k_-$ by its Taylor expansion :

Lemma IV.4 *There exists a constant $K_3 > 0$ such that :*

$$
\left| \partial_+^2 A_{G,\Lambda}(\pi T, 1, 0) - \partial_+^2 \tilde{A}_{G,\Lambda}(\pi T, 1, 0) \right| \leq K_3 . \qquad \text{(IV.52)}
$$

Proof :

$$\partial_+^2 A_{G,\Lambda}(\pi T,1,0)-\partial_+^2\tilde A_{G,\Lambda}(\pi T,1,0)$$
$$=\sum_{\substack{\{\sigma_j\},\ i_j,s_j^+>i_{max}(T)-\Lambda\\ \sigma_1\ \text{right}}}\int d^3x\, x_+^2\left[C_{\sigma_1}(x)\overline{C}_{\sigma_2}(x)\overline{C}_{\sigma_3}(x)-\tilde C_{\sigma_1^r}(x)\overline{\tilde C}_{\sigma_2^\ell}\overline{\tilde C}_{\sigma_3^\ell}(x)\right]e^{-(\pi T x_0+x_+)}$$
$$+2\sum_{\substack{\{\sigma_j\},\ i_j,s_j^+>i_{max}(T)-\Lambda\\ \sigma_2\ \text{right}}}\int d^3x\, x_+^2\left[C_{\sigma_1}(x)\overline{C}_{\sigma_2}(x)\overline{C}_{\sigma_3}(x)-\tilde C_{\sigma_1^\ell}(x)\overline{\tilde C}_{\sigma_2^r}\overline{\tilde C}_{\sigma_3^r}(x)\right]\ , \quad \text{(IV.53)}$$

where

$$\tilde C_{\sigma^r}(x) = \int d^3k\,\frac{u_{\sigma^r}(k)e^{ik.x}}{ik_0+\pi(k_+-1)\cos\frac{\pi}{2}k_-} \quad \text{(IV.54)}$$
$$\tilde C_{\sigma^\ell}(x) = \int d^3k\,\frac{u_{\sigma^l}(k)e^{ik.x}}{ik_0-\pi(k_++1)\cos\frac{\pi}{2}k_-}\ . \quad \text{(IV.55)}$$

Observing that there exists a constant K_4 such that :

$$\left|\cos\frac{\pi}{2}k_+ +\pi(k_+-1)\right| \le K_4(k_+-1)^2 \quad \text{(IV.56)}$$
$$\left|\cos\frac{\pi}{2}k_+ -\pi(k_++1)\right| \le K_4(k_++1)^2 \quad \text{(IV.57)}$$

uniformly in k_+, we have :

$$\left|C_{\sigma^{r(\ell)}}(x)-\tilde C_{\sigma^{r(\ell)}}(x)\right|\le c'.M^{-3s_+-s_-}e^{-c'd_\sigma^\alpha(x)}\ . \quad \text{(IV.58)}$$

Using the relation

$$C_{\sigma_1}\overline{C}_{\sigma_2}\overline{C}_{\sigma_3}-\tilde C_{\sigma_1}\overline{\tilde C}_{\sigma_2}\overline{\tilde C}_{\sigma_3}=(C_{\sigma_1}-\tilde C_{\sigma_1})\overline{C}_{\sigma_2}\overline{C}_{\sigma_3}+\tilde C_{\sigma_1}(\overline{C}_{\sigma_2}-\overline{\tilde C}_{\sigma_2})C_{\sigma_3}+\tilde C_{\sigma_1}\overline{\tilde C}_{\sigma_2}(\overline{C}_{\sigma_3}-\overline{\tilde C}_{\sigma_3})\ , \quad \text{(IV.59)}$$

to create differences of the type $C-\tilde C$, we gain $M^{-2s_+}\le M^{-2(i_{max}-\Lambda)}$ (provided $i_{max}(T)-2\Lambda\ge 0$, which we assume from now on) in the power counting with respect to a single propagator. ∎

At last, we state our main lower bound:

Theorem IV.1 *There exists a constant $K_5>0$ such that :*

$$\left|\partial_+^2\tilde A_G(\pi T,1,0)\right|\ge\frac{K_5}{T}\ . \quad \text{(IV.60)}$$

This theorem with the lemmas of this section obviously imply Theorem II.1, hence the remaining of this paper is devoted to the proof of this Theorem IV.1.

V Integration over $k_{1,+}$, $k_{2,+}$ and $k_{3,+}$

V.1 Approximate expression

We return to equation (III.21), in which all three cutoffs u_Λ have been replaced by 1. Let us write in equation (III.21) $\partial_+^2 \tilde{A}_G(\pi T,1,0)$ as $\partial_+^2 \tilde{A}_{G,1} + 2\partial_+^2 \tilde{A}_{G,2}$ and let us consider the first term $\partial_+^2 \tilde{A}_{G,1}$.

The first propagator (after a change of variable to call the dummy variable q_+ again k_+):

$$\int d^3k_1 \, \frac{e^{ik_1.x} e^{ix_+}}{ik_{1,0} + \pi k_{1,+} \cos\left(\frac{\pi}{2} k_{1,-}\right)} \, . \tag{V.61}$$

For $\cos\left(\frac{\pi}{2}k_{1,-}\right) \neq 0$ we have :

$$\int_{-\infty}^{+\infty} dk_{1,+} \, \frac{e^{ik_{1,+}x_+}}{ik_{1,0} + \pi k_{1,+} \cos\left(\frac{\pi}{2} k_{1,-}\right)} = \frac{1}{\pi \cos\left(\frac{\pi}{2} k_{1,-}\right)} \int_{-\infty}^{+\infty} dk_{1,+} \, \frac{e^{ik_{1,+}x_+}}{k_{1,+} + \left(\frac{ik_{1,0}}{\pi \cos\left(\frac{\pi}{2}k_{1,-}\right)}\right)} \, . \tag{V.62}$$

The corresponding residue is $\exp\left(\frac{k_{1,0}x_+}{\pi\cos\left(\frac{\pi}{2}k_{1,-}\right)}\right)$. If $x_+ > 0$, then we move the path of integration upwards. It is oriented in the positive direction, so we get :

$$\chi(x_+ > 0)\, \chi\left(-\frac{k_{1,0}}{\pi\cos\left(\frac{\pi}{2}k_{1,-}\right)} > 0\right) 2i\pi \exp\left(\frac{k_{1,0}x_+}{\pi\cos(\frac{\pi}{2}k_{1,-})}\right) \, . \tag{V.63}$$

If $x_+ < 0$, then the path of integration is moved downwards, and we get a minus sign owing to the negative direction. Hence:

$$\begin{aligned} \int_{-\infty}^{+\infty} dk_{1,+} \, \frac{e^{ik_{1,+}x_+}}{ik_{1,0} + \pi k_{1,+} \cos\left(\frac{\pi}{2} k_{1,-}\right)} &= \frac{2i}{\cos\left(\frac{\pi}{2}k_{1,-}\right)} \exp\left(\frac{k_{1,0}x_+}{\pi\cos\left(\frac{\pi}{2}k_{1,-}\right)}\right) \\ &\left[\chi(x_+ > 0)\, \chi\left(-\frac{k_{1,0}}{\pi\cos\left(\frac{\pi}{2}k_{1,-}\right)} > 0\right) - \chi(x_+ < 0)\, \chi\left(-\frac{k_{1,0}}{\pi\cos\left(\frac{\pi}{2}k_{1,-}\right)} < 0\right)\right] \, . \end{aligned} \tag{V.64}$$

We treat analogously the integrations over $k_{2,+}$ and $k_{3,+}$. The only difference with the previous case is that these propagators were near the left singularity $k_+ \simeq -1$, so there are some sign changes in $q_{2,+}$ and $q_{3,+} \approx -1$. We obtain :

$$\begin{aligned} \int_{-\infty}^{+\infty} dk_{2,+} \, \frac{e^{ik_{2,+}x_+}}{-ik_{2,0} - \pi k_{2,+} \cos\left(\frac{\pi}{2} k_{2,-}\right)} &= \frac{-2i}{\cos\left(\frac{\pi}{2}k_{2,-}\right)} \exp\left(-\frac{k_{2,0}x_+}{\pi\cos(\frac{\pi}{2}k_{2,-})}\right) \\ &\left[\chi(x_+ > 0)\, \chi\left(\frac{k_{2,0}}{\pi\cos\left(\frac{\pi}{2}k_{2,-}\right)} < 0\right) - \chi(x_+ < 0)\, \chi\left(\frac{k_{2,0}}{\pi\cos\left(\frac{\pi}{2}k_{2,-}\right)} > 0\right)\right] \, . \end{aligned} \tag{V.65}$$

$$\partial_+^2 \tilde{A}_{G,1}(\pi T,1,0) = -8i \int d^3x \int dk_{1,0}\, dk_{1,-}\, dk_{2,0}\, dk_{2,-}\, dk_{3,0}\, dk_{3,-}$$

$$x_+^2 \frac{\exp\left(\left(\frac{k_{1,0}}{\pi\cos\left(\frac{\pi}{2}k_{1,-}\right)} + \frac{k_{2,0}}{\pi\cos\left(\frac{\pi}{2}k_{2,-}\right)} + \frac{k_{3,0}}{\pi\cos\left(\frac{\pi}{2}k_{3,-}\right)}\right)x_+\right)}{\cos\left(\frac{\pi}{2}k_{1,-}\right)\cos\left(\frac{\pi}{2}k_{2,-}\right)\cos\left(\frac{\pi}{2}k_{3,-}\right)} e^{i(k_{1,0}+k_{2,0}+k_{3,0}+\pi T)x_0}\, e^{i(k_{1,-}+k_{2,-}+k_{3,-})x_-}$$

$$\left[\chi(x_+>0)\,\chi\left(\frac{k_{1,0}}{\pi\cos\left(\frac{\pi}{2}k_{1,-}\right)}<0\right)\chi\left(\frac{k_{2,0}}{\pi\cos\left(\frac{\pi}{2}k_{2,-}\right)}<0\right)\chi\left(\frac{k_{3,0}}{\pi\cos\left(\frac{\pi}{2}k_{3,-}\right)}<0\right)\right.$$

$$\left.-\,\chi(x_+<0)\,\chi\left(\frac{k_{1,0}}{\pi\cos\left(\frac{\pi}{2}k_{1,-}\right)}>0\right)\chi\left(\frac{k_{2,0}}{\pi\cos\left(\frac{\pi}{2}k_{2,-}\right)}>0\right)\chi\left(\frac{k_{3,0}}{\pi\cos\left(\frac{\pi}{2}k_{3,-}\right)}>0\right)\right]. \tag{V.66}$$

V.2 Integration over x_0 and $k_{3,0}$

The calculation is done integrating over x_0, which leads to a delta function in the integrand, denoted with a slight abuse of notation by $\delta(k_{1,0}+k_{2,0}+k_{3,0}+\pi T=0)$. In fact, there is a prefactor $\frac{1}{T}$ that compensates the T factor of $\int dk_{3,0}$: remember that $\int dk_{3,0}$ means precisely : $2\pi T\sum_{k_{3,0}\in\pi T+2\pi T\mathbb{Z}}$. This yields :

$$\partial_+^2 \tilde{A}_{G,1}(\pi T,1,0) = -8i \int dx_+\, dx_- \int dk_{1,0}\, dk_{1,-}\, dk_{2,0}\, dk_{2,-}\, dk_{3,0}\, dk_{3,-}$$

$$x_+^2 \frac{e^{\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}+\frac{k_{3,0}}{\pi\cos(\frac{\pi}{2}k_{3,-})}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}k_{3,-})} e^{i(k_{1,-}+k_{2,-}+k_{3,-})x_-}\,\delta(k_{1,0}+k_{2,0}+k_{3,0}+\pi T=0)$$

$$\left[\chi(x_+>0)\,\chi(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}<0)\,\chi(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}<0)\chi(\frac{k_{3,0}}{\pi\cos(\frac{\pi}{2}k_{3,-})}<0)\right.$$

$$\left.-\,\chi(x_+<0)\,\chi(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}>0)\,\chi(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}>0)\chi(\frac{k_{3,0}}{\pi\cos(\frac{\pi}{2}k_{3,-})}>0)\right]. \tag{V.67}$$

At this stage, we can use the delta function to integrate, for instance, over $k_{3,0}$:

$$\partial_+^2 \tilde{A}_{G,1}(\pi T,1,0) = -8i \int dx_+\, dx_- \int dk_{1,0}\, dk_{1,-}\, dk_{2,0}\, dk_{2,-}\, dk_{3,-}$$

$$x_+^2 \frac{e^{\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}-\frac{k_{1,0}+k_{2,0}+\pi T}{\pi\cos(\frac{\pi}{2}k_{3,-})}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}k_{3,-})} e^{i(k_{1,-}+k_{2,-}+k_{3,-})x_-}$$

$$\left[\chi(x_+>0)\,\chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}<0\right)\chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}<0\right)\chi\left(\frac{k_{1,0}+k_{2,0}+\pi T}{\pi\cos(\frac{\pi}{2}k_{3,-})}>0\right)\right.$$

$$\left.-\,\chi(x_+<0)\,\chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}>0\right)\chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}>0\right)\chi\left(\frac{k_{1,0}+k_{2,0}+\pi T}{\pi\cos(\frac{\pi}{2}k_{3,-})}<0\right)\right]. \tag{V.68}$$

V.3 Simplification

This rather complicated expression can be slightly simplified. Indeed, if we perform the change of variables :

$$\left\{ \begin{array}{rcl} x'_+ &=& -x_+ \\ k'_{1,0} &=& -k_{1,0} \\ k'_{2,0} &=& -k_{2,0} \end{array} \right. \tag{V.69}$$

the integral

$$\int dx_+\, dx_- \int dk_{1,0}\, dk_{1,-}\, dk_{2,0}\, dk_{2,-}\, dk_{3,-}\; x_+^2 \, \frac{e^{\left(\frac{k_{1,0}}{\pi \cos(\frac{\pi}{2}k_{1,-})} + \frac{k_{2,0}}{\pi \cos(\frac{\pi}{2}k_{2,-})} - \frac{k_{1,0}+k_{2,0}+\pi T}{\pi \cos(\frac{\pi}{2}k_{3,-})} \right) x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}k_{3,-})}$$
$$\chi(x_+ < 0)\, \chi\left(\frac{k_{1,0}}{\pi \cos(\frac{\pi}{2}k_{1,-})} > 0 \right) \chi\left(\frac{k_{2,0}}{\pi \cos(\frac{\pi}{2}k_{2,-})} > 0 \right) \chi\left(\frac{k_{1,0}+k_{2,0}+\pi T}{\pi \cos(\frac{\pi}{2}k_{3,-})} < 0 \right) \tag{V.70}$$

becomes :

$$\int dx'_+\, dx_- \int dk'_{1,0}\, dk_{1,-}\, dk'_{2,0}\, dk_{2,-}\, dk_{3,-}\; x'^2_+ \, \frac{e^{\left(\frac{k'_{1,0}}{\pi \cos(\frac{\pi}{2}k_{1,-})} + \frac{k'_{2,0}}{\pi \cos(\frac{\pi}{2}k_{2,-})} - \frac{k'_{1,0}+k'_{2,0}-\pi T}{\pi \cos(\frac{\pi}{2}k_{3,-})} \right) x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}k_{3,-})}$$
$$\chi(x'_+ > 0)\, \chi\left(\frac{k'_{1,0}}{\pi \cos(\frac{\pi}{2}k_{1,-})} < 0 \right) \chi\left(\frac{k'_{2,0}}{\pi \cos(\frac{\pi}{2}k_{2,-})} < 0 \right) \chi\left(\frac{k'_{1,0}+k'_{2,0}-\pi T}{\pi \cos(\frac{\pi}{2}k_{3,-})} > 0 \right) . \tag{V.71}$$

Consequently the previous expression of $\partial_+^2 \tilde{A}_{G,1}(\pi T, 1, 0)$ can be factorized :

$$\partial_+^2 \tilde{A}_{G,1}(\pi T,1,0) = -8i \int dx_+\, dx_- \int dk_{1,0}\, dk_{1,-}\, dk_{2,0}\, dk_{2,-}\, dk_{3,-}$$
$$x_+^2 \, \frac{e^{\left(\frac{k_{1,0}}{\pi \cos(\frac{\pi}{2}k_{1,-})} + \frac{k_{2,0}}{\pi \cos(\frac{\pi}{2}k_{2,-})} - \frac{k_{1,0}+k_{2,0}}{\pi \cos(\frac{\pi}{2}k_{3,-})} \right) x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}k_{3,-})} \, e^{i(k_{1,-}+k_{2,-}+k_{3,-})x_-}$$
$$\chi(x_+ > 0)\, \chi\left(\frac{k_{1,0}}{\pi \cos(\frac{\pi}{2}k_{1,-})} < 0 \right) \chi\left(\frac{k_{2,0}}{\pi \cos(\frac{\pi}{2}k_{2,-})} < 0 \right)$$
$$\left[e^{\frac{-T x_+}{\cos(\frac{\pi}{2}(k_{3,-}))}} \chi\left(\frac{k_{1,0}+k_{2,0}+\pi T}{\pi \cos(\frac{\pi}{2}k_{3,-})} > 0 \right) - e^{\frac{T x_+}{\cos(\frac{\pi}{2}(k_{3,-}))}} \chi\left(\frac{k_{1,0}+k_{2,0}-\pi T}{\pi \cos(\frac{\pi}{2}k_{3,-})} > 0 \right) \right] . \tag{V.72}$$

VI Integration over x_- and $k_{3,-}$

We now are going to perform the integration over x_-, which will provide a conservation rule for the moments $k_{1,0}$, $k_{2,0}$ and $k_{3,0}$, but only modulo 2. To understand that, remember that $\int dx_+\, dx_-$ means more precisely : $\sum'_{(x_+,x_-)\in\left(\frac{\pi}{2}\mathbb{Z}\right)^2}$, where the prime in the sum means that one has to respect a parity condition between x_+ and x_-. By slight abuse of language, we say that x_+ and x_- have

the same parity when $x_+ + x_- \in \pi\mathbb{Z}$. So $\sum'_{(x_+,x_-)\in\left(\frac{\pi}{2}\mathbb{Z}\right)^2}$ does not mean : $\sum_{x_+\in\frac{\pi}{2}\mathbb{Z}}\sum_{x_-\in\frac{\pi}{2}\mathbb{Z}}$ but $\sum_{x_+\in\pi\mathbb{Z}}\sum_{x_-\in\pi\mathbb{Z}} + \sum_{x_+\in\frac{\pi}{2}+\pi\mathbb{Z}}\sum_{x_-\in\frac{\pi}{2}+\pi\mathbb{Z}}$. Now,

$$\sum_{x_-\in\pi\mathbb{Z}} e^{i(k_{1,-}+k_{2,-}+k_{3,-})x_-} = \delta(k_{1,-}+k_{2,-}+k_{3,-}=0[2]) \tag{VI.73}$$

where by $\delta(k_{1,-}+k_{2,-}+k_{3,-}=0[2])$, we denote : $\sum_{n\in\mathbb{Z}}\delta(k_{1,-}+k_{2,-}+k_{3,-}=2n)$. Then it is clear that

$$\sum_{x_-\in\frac{\pi}{2}+\pi\mathbb{Z}} e^{i(k_{1,-}+k_{2,-}+k_{3,-})x_-} = e^{i\frac{\pi}{2}(k_{1,-}+k_{2,-}+k_{3,-})}\,\delta(k_{1,-}+k_{2,-}+k_{3,-}=0[2])\ . \tag{VI.74}$$

Indeed, the factor $e^{i\frac{\pi}{2}(k_{1,-}+k_{2,-}+k_{3,-})}$ can take only two values : 1 if $k_{1,-}+k_{2,-}+k_{3,-}=0[4]$, and -1 if $k_{1,-}+k_{2,-}+k_{3,-}=2[4]$. Hence it is convenient to distinguish these two cases and write :

$$\delta(k_{1,-}+k_{2,-}+k_{3,-}=0[2]) = \delta(k_{1,-}+k_{2,-}+k_{3,-}=0[4]) + \delta(k_{1,-}+k_{2,-}+k_{3,-}=2[4]) \tag{VI.75}$$

and

$$\begin{aligned} &e^{i\frac{\pi}{2}(k_{1,-}+k_{2,-}+k_{3,-})}\delta(k_{1,-}+k_{2,-}+k_{3,-}=0[2]) \\ &= \delta(k_{1,-}+k_{2,-}+k_{3,-}=0[4]) - \delta(k_{1,-}+k_{2,-}+k_{3,-}=2[4])\ . \end{aligned} \tag{VI.76}$$

At this stage, we can gather the previous remarks in the following formula :

$$\begin{aligned} \partial_+^2\tilde{A}_{G,1}(\pi T,1,0) &= -8i\sum_{x_+\in\frac{\pi}{2}\mathbb{N}^*}\int dk_{1,0}\,dk_{2,0}\,dk_{1,-}\,dk_{2,-}\,dk_{3,-} \\ &x_+^2\,\frac{e^{\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}-\frac{k_{1,0}+k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{3,-})}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}k_{3,-})}\,\delta(k_{1,-}+k_{2,-}+k_{3,-}=0[4]) \\ &\chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}<0\right)\chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}<0\right) \\ &\left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{3,-}))}}\chi\left(\frac{k_{1,0}+k_{2,0}+\pi T}{\pi\cos(\frac{\pi}{2}k_{3,-})}>0\right) - e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{3,-}))}}\chi\left(\frac{k_{1,0}+k_{2,0}-\pi T}{\pi\cos(\frac{\pi}{2}k_{3,-})}>0\right)\right] \\ &-8i\sum_{x_+\in\frac{\pi}{2}\mathbb{N}^*}\int dk_{1,0}\,dk_{2,0}\,dk_{1,-}\,dk_{2,-}\,dk_{3,-} \\ &x_+^2\,\frac{e^{\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}-\frac{k_{1,0}+k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{3,-})}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}k_{3,-})}\,\delta(k_{1,-}+k_{2,-}+k_{3,-}=2[4]) \\ &[\chi(x_+\ even)-\chi(x_+\ odd)]\chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}<0\right)\chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}<0\right) \\ &\left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{3,-}))}}\chi\left(\frac{k_{1,0}+k_{2,0}+\pi T}{\pi\cos(\frac{\pi}{2}k_{3,-})}>0\right) - e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{3,-}))}}\chi\left(\frac{k_{1,0}+k_{2,0}-\pi T}{\pi\cos(\frac{\pi}{2}k_{3,-})}>0\right)\right]\ . \end{aligned} \tag{VI.77}$$

Then we can perform the integration over $k_{3,-}$. Formally, we only need to replace $\cos(\frac{\pi}{2}k_{3,-})$ by $\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))$ for the first piece and with $-\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))$ for the second piece. We obtain :

$$
\begin{aligned}
\partial_+^2 \tilde{A}_{G,1}(\pi T,1,0) = & -8i \sum_{x_+\in\frac{\pi}{2}\mathbb{N}^*} \int dk_{1,0}\, dk_{2,0}\, dk_{1,-}\, dk_{2,-} \\
& x_+^2 \frac{e^{\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}-\frac{k_{1,0}+k_{2,0}}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))} \chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}<0\right)\chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}<0\right) \\
& \Bigg[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}} \chi\left(\frac{k_{1,0}+k_{2,0}+\pi T}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}>0\right) \\
& - e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}} \chi\left(\frac{k_{1,0}+k_{2,0}-\pi T}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}>0\right)\Bigg] \\
& +8i \sum_{x_+\in\frac{\pi}{2}\mathbb{N}^*} \int dk_{1,0}\, dk_{2,0}\, dk_{1,-}\, dk_{2,-} x_+^2 \frac{e^{\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}+\frac{k_{1,0}+k_{2,0}}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))} \\
& [\chi(x_+\ even)-\chi(x_+\ odd)]\chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}<0\right)\chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}<0\right) \\
& \Bigg[e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}} \chi\left(\frac{k_{1,0}+k_{2,0}+\pi T}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}<0\right) \\
& - e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}} \chi\left(\frac{k_{1,0}+k_{2,0}-\pi T}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}<0\right)\Bigg]. \qquad \text{(VI.78)}
\end{aligned}
$$

Now it is clear that $\partial_+^2 \tilde{A}_{G,1}(\pi T,1,0)$ is a purely imaginary number. The first piece gives the leading behavior as $T\to 0$. Indeed the second piece is much smaller, thanks to the compensation in $[\chi(x_+\ even)-\chi(x_+\ odd)]$. Indeed the sum

$$\sum_{x_+\in\frac{\pi}{2}\mathbb{N}^*} x_+^2[\chi(x_+\ even)-\chi(x_+\ odd)]... \qquad \text{(VI.79)}$$

can be written as a sum of two terms of the type

$$\sum_{n\in\mathbb{N}^*}\int dk e^{-2A(k)n}[(2n)^2-(2n+1)^2e^{-A(k)}]B(k) \qquad \text{(VI.80)}$$

where A and B are independent of n and $A(k)>0$. Then we can decompose the remaining integrals $\int dk$ into two zones, according to whether $A(k)\geq T^{1/3}$ or $A(k)\leq T^{1/3}$. In the first zone we do not need to exploit the subtraction, but we have simply $\sum_{n\in\mathbb{N}^*} n^2 e^{-2T^{1/3}n}\leq c.T^{-2/3} << T^{-1}$, and in the second zone, we use $|(2n)^2-(2n+1)^2e^{-A(k)}|\leq 4n+1+(2n+1)^2A(k)\leq 4n+1+(2n+1)^2T^{1/3}$. The first term in $4n+1$ is then bounded with the same techniques than Lemma

IV.1, but the factor $M^{\inf\{i_j\}+3\inf\{s_j^+\}+\inf\{s_j^-\}}$ is replaced by $M^{\inf\{i_j\}+2\inf\{s_j^+\}+\inf\{s_j^-\}}$ and the bound corresponding to equation (IV.40) gives now

$$\sum_{\{i_j\},\{s_j^+\},\{s_j^-\}} M^{-\frac{1}{3}\Delta\{i_j\}} M^{-\frac{2}{3}\Delta\{s_j^+\}} M^{-\frac{1}{3}\Sigma s_j^-} \leq 0(1) \;, \tag{VI.81}$$

hence this piece does not diverge at all when $T \to 0$. Finally the piece with the factor $(2n+1)^2 T^{1/3}$ is similar to previous pieces, except for the new factor $T^{1/3}$, so that it is bounded in the manner of Lemma IV.1 by a factor $c.T^{-1}T^{1/3} = c.T^{-2/3}$.

So we are left to study :

$$\begin{aligned} A_1(T) = -8i \sum_{x_+ \in \frac{\pi}{2}\mathbb{N}^*} \int dk_{1,0}\, dk_{2,0}\, dk_{1,-}\, dk_{2,-} x_+^2 \frac{e^{\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})} + \frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})} - \frac{k_{1,0}+k_{2,0}}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))} \\ \chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})} < 0\right)\chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})} < 0\right) \\ \left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi\left(\frac{k_{1,0}+k_{2,0}+\pi T}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))} > 0\right)\right. \\ \left. - e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi\left(\frac{k_{1,0}+k_{2,0}-\pi T}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))} > 0\right)\right] . \end{aligned} \tag{VI.82}$$

VII Leading contribution

VII.1 Symmetry properties

Henceforward, we shall denote the integrand by $F(x_+, k_{1,0}, k_{2,0}, k_{1,-}, k_{2,-})$ so that :

$$A_1(T) = -8i \sum_{x_+ \in \frac{\pi}{2}\mathbb{N}^*} \int dk_{1,0}\, dk_{2,0}\, dk_{1,-}\, dk_{2,-} F(x_+, k_{1,0}, k_{2,0}, k_{1,-}, k_{2,-}) \;. \tag{VII.83}$$

The couple of variables of integration $(k_{1,-}, k_{2,-})$ describes the square $[-2,2]^2$. To pursue the calculation, we shall make a partition of $[-2,2]^2$, according to the signs of $\cos(\frac{\pi}{2}k_{1,-})$, $\cos(\frac{\pi}{2}k_{2,-})$ and $\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))$. This partition is represented in Figure 4:

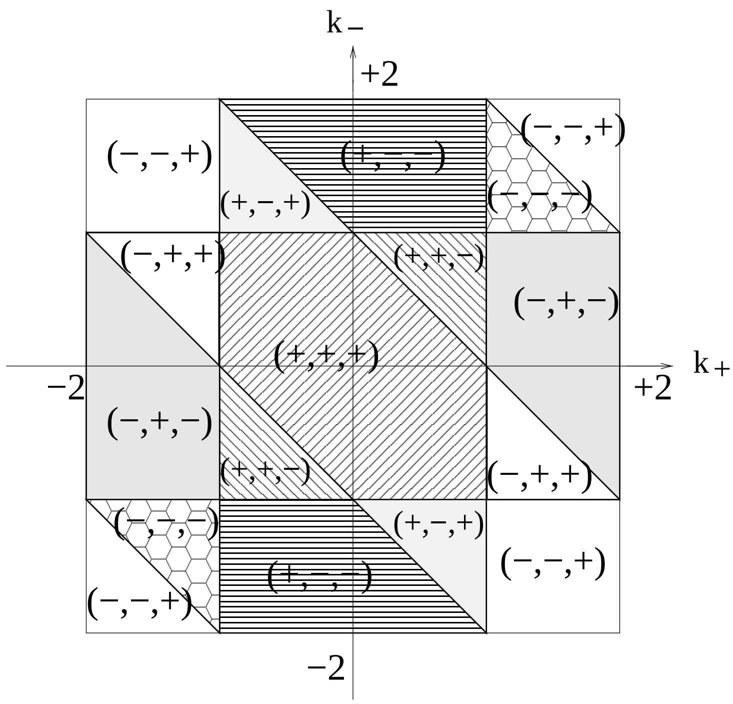

Figure 4: The domain of integration in (k_+, k_-)

The signs of the three cosines determine eight cases we can discuss separately. In fact, it is possible to restrict the domain of integration thanks to symmetries of the integrand involving the variables $k_{1,-}$ and $k_{2,-}$ together with the variables $k_{1,0}$ and $k_{2,0}$, which describe independently the set $\pi T + 2\pi T\mathbb{Z}$.

It is evident, by the parity of the cosine function, that the integrand is invariant under the replacement $k_{1,-} \to -k_{1,-}$ and $k_{2,-} \to -k_{2,-}$, which corresponds to the central symmetry with respect to the origin (0,0). Hence we have :

$$A_1(T) = -16i \sum_{x_+ \in \frac{\pi}{2}\mathbb{N}^*} \int dk_{1,0}\, dk_{2,0} \int_{-2}^{2} dk_{1,-} \int_{0}^{2} dk_{2,-}\, F(x_+, k_{1,0}, k_{2,0}, k_{1,-}, k_{2,-}) \ . \qquad \text{(VII.84)}$$

Symmetry properties of $F(x_+, k_{1,0}, k_{2,0}, k_{1,-}, k_{2,-})$ can be exploited further. The above integral

may be separated into two pieces :

$$A_1(T) = -16i \left(\sum_{x_+ \in \frac{\pi}{2}\mathbb{N}^*} \int dk_{1,0}\, dk_{2,0} \int_{-2}^{0} dk_{1,-} \int_{0}^{2} dk_{2,-}\, F(x_+, k_{1,0}, k_{2,0}, k_{1,-}, k_{2,-}) \right.$$
$$\left. + \sum_{x_+ \in \frac{\pi}{2}\mathbb{N}^*} \int dk_{1,0}\, dk_{2,0} \int_{0}^{2} dk_{1,-} \int_{0}^{2} dk_{2,-}\, F(x_+, k_{1,0}, k_{2,0}, k_{1,-}, k_{2,-}) \right) . \quad \text{(VII.85)}$$

For the first integral, one can easily verify that the integrand $F(x_+, k_{1,0}, k_{2,0}, k_{1,-}, k_{2,-})$ is invariant under the change of variables :

$$k'_{1,0} = k_{2,0}\ ,\ k'_{2,0} = k_{1,0}\ ,\ k'_{1,-} = -k_{2,-}\ ,\ k'_{2,-} = -k_{1,-}\ . \quad \text{(VII.86)}$$

We get :

$$\int dk_{1,0}\, dk_{2,0} \int_{-2}^{0} dk_{1,-} \int_{0}^{2} dk_{2,-}\, F(x_+, k_{1,0}, k_{2,0}, k_{1,-}, k_{2,-}) =$$
$$2 \int dk_{1,0}\, dk_{2,0} \int_{-2}^{0} dk_{1,-} \int_{-k_{1,-}}^{2} dk_{2,-}\, F(x_+, k_{1,0}, k_{2,0}, k_{1,-}, k_{2,-}) . \quad \text{(VII.87)}$$

We treat analogously the other integral ; we set :

$$k'_{1,0} = k_{2,0}\ ,\ k'_{2,0} = k_{1,0}\ ,\ k'_{1,-} = k_{2,-}\ ,\ k'_{2,-} = k_{1,-}\ . \quad \text{(VII.88)}$$

Hence :

$$\int dk_{1,0}\, dk_{2,0} \int_{0}^{2} dk_{1,-} \int_{0}^{2} dk_{2,-}\, F(x_+, k_{1,0}, k_{2,0}, k_{1,-}, k_{2,-}) =$$
$$2 \int dk_{1,0}\, dk_{2,0} \int_{0}^{2} dk_{1,-} \int_{k_{1,-}}^{2} dk_{2,-}\, F(x_+, k_{1,0}, k_{2,0}, k_{1,-}, k_{2,-}) . \quad \text{(VII.89)}$$

Finally, we have established owing to symmetry properties that :

$$A_1(T) = -32i \int dk_{1,0}\, dk_{2,0} \iint_{\mathscr{T}} dk_{1,-}\, dk_{2,-}\, F(x_+, k_{1,0}, k_{2,0}, k_{1,-}, k_{2,-})\ , \quad \text{(VII.90)}$$

the domain of integration in $(k_{1,-}, k_{2,-})$ being the triangle $\mathscr{T}$ delimited by the lines $k_{2,-} = 2$, $k_{2,-} = k_{1,-}$ and $k_{2,-} = -k_{1,-}$.

VII.2 Discussion of the various cases

VII.2.1 The $(+,+,+)$ case

As we have said, it is now convenient to carry a discussion about the signs of $\cos(\frac{\pi}{2}k_{1,-})$, $\cos(\frac{\pi}{2}k_{2,-})$ and $\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))$, which allows us to perform explicitly the summation over $k_{1,0}$ and $k_{2,0}$ in each case.

We first begin with the case :

$$\left\{\begin{array}{rcl}\cos(\frac{\pi}{2}k_{1,-}) & > & 0\\ \cos(\frac{\pi}{2}k_{2,-}) & > & 0\\ \cos(\frac{\pi}{2}(k_{1,-}+k_{2,-})) & > & 0\end{array}\right. , \tag{VII.91}$$

that we will denote as $(+,+,+)$. The corresponding contribution to $A_1(T)$ is :

$$\begin{aligned}A_1^{(+,+,+)}(T) = -32i \sum_{x_+\in\frac{\pi}{2}\mathbb{N}^*}\int dk_{1,0}\,dk_{2,0} \iint_{\mathscr{T}_{(+,+,+)}} dk_{1,-}\,dk_{2,-}\\ x_+^2\,\frac{e^{\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}-\frac{k_{1,0}+k_{2,0}}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\,\chi(k_{1,0}<0)\,\chi(k_{2,0}<0)\\ \left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(k_{1,0}+k_{2,0}>-\pi T)-e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(k_{1,0}+k_{2,0}>\pi T)\right] ,\end{aligned} \tag{VII.92}$$

where $\mathscr{T}_{(+,+,+)}$ denotes the subset of $\mathscr{T}$ where the signs of the cosines are $(+,+,+)$ respectively. Since the conditions $k_{1,0}<0$, $k_{2,0}<0$ and $k_{1,0}+k_{2,0}>\pm\pi T$ are incompatible, $A_1^{(+,+,+)}=0$.

VII.2.2 The $(+,+,-)$ case

Let us consider the case :

$$\left\{\begin{array}{rcl}\cos(\frac{\pi}{2}k_{1,-}) & > & 0\\ \cos(\frac{\pi}{2}k_{2,-}) & > & 0\\ \cos(\frac{\pi}{2}(k_{1,-}+k_{2,-})) & < & 0\end{array}\right. , \tag{VII.93}$$

corresponding to the sign configuration $(+,+,-)$. We have :

$$\begin{aligned}A_1^{(+,+,-)}(T) = -32i \sum_{x_+\in\frac{\pi}{2}\mathbb{N}^*}\int dk_{1,0}\,dk_{2,0} \iint_{\mathscr{T}_{(+,+,-)}} dk_{1,-}\,dk_{2,-}\\ x_+^2\,\frac{e^{\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}-\frac{k_{1,0}+k_{2,0}}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\,\chi(k_{1,0}<0)\,\chi(k_{2,0}<0)\\ \left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(k_{1,0}+k_{2,0}<-\pi T)-e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(k_{1,0}+k_{2,0}<\pi T)\right] .\end{aligned} \tag{VII.94}$$

The conditions $\chi(k_{1,0}+k_{2,0}<\pm\pi T)$ can obviously be omitted. We must compute the following expression :

$$\begin{aligned}\sum_{n=0}^{+\infty}\sum_{p=0}^{+\infty} e^{-(2n+1)\left(\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}-\frac{1}{\cos(\frac{\pi}{2}k_{1,-}+k_{2,-})}\right)Tx_+}\, e^{-(2p+1)\left(\frac{1}{\cos(\frac{\pi}{2}k_{2,-})}-\frac{1}{\cos(\frac{\pi}{2}k_{1,-}+k_{2,-})}\right)Tx_+}\\ \left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}-e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\right] ,\end{aligned} \tag{VII.95}$$

which gives :

$$\frac{e^{-\left(\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}+\frac{1}{\cos(\frac{\pi}{2}k_{2,-})}-\frac{1}{\cos(\frac{\pi}{2}k_{1,-}+k_{2,-})}\right)Tx_+}\left[1-e^{\frac{2Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\right]}{\left[1-e^{-2\left(\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}-\frac{1}{\cos(\frac{\pi}{2}k_{1,-}+k_{2,-})}\right)Tx_+}\right]\left[1-e^{-2\left(\frac{1}{\cos(\frac{\pi}{2}k_{2,-})}-\frac{1}{\cos(\frac{\pi}{2}k_{1,-}+k_{2,-})}\right)Tx_+}\right]}\ . \tag{VII.96}$$

This is clearly a positive real number, and therefore we conclude that

$$iA_1^{(+,+,-)}(T)\leq 0\ . \tag{VII.97}$$

Indeed, the minus sign of the prefactor $-32i$ is compensated by the minus sign of the product $\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}k_{1,-}+k_{2,-})$.

VII.2.3 The $(+,-,+)$ case

We now consider the $(+,-,+)$ case. The corresponding contribution writes :

$$\begin{aligned}A_1^{(+,-,+)}(T)=&-32i\sum_{x_+\in\frac{\pi}{2}\mathbb{N}^*}\int dk_{1,0}\,dk_{2,0}\iint_{\mathscr{T}_{(+,-,+)}}dk_{1,-}\,dk_{2,-}\\ &x_+^2\frac{e^{\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}-\frac{k_{1,0}+k_{2,0}}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\chi(k_{1,0}<0)\,\chi(k_{2,0}>0)\\ &\left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(k_{1,0}+k_{2,0}>-\pi T)-e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(k_{1,0}+k_{2,0}>\pi T)\right]\ .\end{aligned} \tag{VII.98}$$

Here like in all the other cases, we have to sum geometric sequences whose ratio is explicitly strictly smaller than 1. This facilitates the discussion of the signs of the corresponding quantities, as we shall see. If we perform the summation over $k_{1,0}$, we are lead to a geometric sequence whose ratio is $e^{-2\left(\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}-\frac{1}{\cos(\frac{\pi}{2}k_{1,-}+k_{2,-})}\right)}$, which will lead to a factor $\left[1-e^{-2\left(\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}-\frac{1}{\cos(\frac{\pi}{2}k_{1,-}+k_{2,-})}\right)}\right]^{-1}$ whose sign is not uniform in $(k_{1,-},k_{2,-})$.

Consequently we introduce the variable $s=k_{1,0}+k_{2,0}$ and replace $k_{2,0}$ by $s-k_{1,0}$. We must compute :

$$\begin{aligned}&\int dk_{1,0}\,ds\,e^{\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{s-k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}-\frac{s}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)x_+}\chi(k_{1,0}<0)\chi(s>k_{1,0})\\ &\qquad\left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(s>-\pi T)-e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(s>\pi T)\right]\ .\end{aligned} \tag{VII.99}$$

The variable s describes the set $2\pi T\mathbb{Z}$ and the condition $\chi(s>k_{1,0})$ can be omitted. Thus the previous expression writes :

$$\sum_{n=0}^{+\infty} e^{-(2n+1)\left(\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}-\frac{1}{\cos(\frac{\pi}{2}k_{2,-})}\right)} \left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}} \sum_{p=0}^{+\infty} e^{-2p\left(\frac{1}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}-\frac{1}{\cos(\frac{\pi}{2}k_{2,-})}\right)Tx_+} \right.$$
$$\left. - e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}} \sum_{p=0}^{+\infty} e^{-2p\left(\frac{1}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}-\frac{1}{\cos(\frac{\pi}{2}k_{2,-})}\right)Tx_+} \right] \quad \text{(VII.100)}$$

which is equal to :

$$\frac{e^{-\left(\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}-\frac{1}{\cos(\frac{\pi}{2}k_{2,-})}+\frac{1}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)Tx_+}\left[1-e^{\frac{2Tx_+}{\cos(\frac{\pi}{2}k_{2,-})}}\right]}{\left[1-e^{-2\left(\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}-\frac{1}{\cos(\frac{\pi}{2}k_{2,-})}\right)Tx_+}\right]\left[1-e^{-2\left(\frac{1}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}-\frac{1}{\cos(\frac{\pi}{2}k_{2,-})}\right)Tx_+}\right]} . \quad \text{(VII.101)}$$

This quantity is positive, thus the conclusion follows :

$$iA_1^{(+,-,+)}(T) \leq 0 . \quad \text{(VII.102)}$$

VII.2.4 The $(+,-,-)$ case

Let us examine now the $(+,-,-)$ case. The contribution is :

$$A_1^{(+,-,-)}(T) = -32i \sum_{x_+\in\frac{\pi}{2}\mathbb{N}^*} \int dk_{1,0}\,dk_{2,0} \iint_{\mathscr{T}_{(+,-,-)}} dk_{1,-}\,dk_{2,-}$$
$$x_+^2 \frac{e^{\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}-\frac{k_{1,0}+k_{2,0}}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\chi(k_{1,0}<0)\,\chi(k_{2,0}>0)$$
$$\left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(k_{1,0}+k_{2,0}<-\pi T) - e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(k_{1,0}+k_{2,0}<\pi T)\right] . \quad \text{(VII.103)}$$

We set $k_{1,0} = s - k_{2,0}$ and we compute :

$$\int ds\,dk_{2,0}\,e^{\left(\frac{s-k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}-\frac{s}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)x_+}\chi(s<k_{2,0})\chi(k_{2,0}>0)$$
$$\left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(s<-\pi T) - e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(s<\pi T)\right] . \quad \text{(VII.104)}$$

The condition $\chi(s<k_{2,0})$ may be omitted and we must evaluate :

$$\sum_{n=0}^{+\infty} e^{(2n+1)\left(\frac{1}{\cos(\frac{\pi}{2}k_{2,-})}-\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}\right)Tx_+} \left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}} \sum_{p=1}^{+\infty} e^{-2p\left(\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}-\frac{1}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)Tx_+} - e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}} \sum_{p=0}^{+\infty} e^{-2p\left(\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}-\frac{1}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)Tx_+} \right] . \quad \text{(VII.105)}$$

We find :

$$\frac{e^{-\left(\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}-\frac{1}{\cos(\frac{\pi}{2}k_{2,-})}-\frac{1}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)Tx_+}\left[e^{\frac{-2Tx_+}{\cos(\frac{\pi}{2}k_{1,-})}}-1\right]}{\left[1-e^{-2\left(\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}-\frac{1}{\cos(\frac{\pi}{2}k_{2,-})}\right)Tx_+}\right]\left[1-e^{-2\left(\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}-\frac{1}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)Tx_+}\right]} . \quad \text{(VII.106)}$$

This is a negative number, therefore

$$iA_1^{(+,-,-)}(T) \leq 0 . \quad \text{(VII.107)}$$

VII.2.5 The $(-,+,+)$ and (-,+,-) cases

There is no discussion to carry out : in fact, for $(k_{1,-},k_{2,-})\in\mathscr{T}$, we have never $\cos(\frac{\pi}{2}k_{1,-})<0$, $\cos(\frac{\pi}{2}k_{2,-})>0$ and $\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))<0$ simultaneously. We also conclude in the same way for the $(-,+,-)$ case.

VII.2.6 The (-,-,+) case

$$A_1^{(-,-,+)}(T) = -32i \sum_{x_+\in\frac{\pi}{2}\mathbb{N}^*} \int dk_{1,0}\, dk_{2,0} \iint_{\mathscr{T}_{(-,-,+)}} dk_{1,-}\, dk_{2,-} \, x_+^2 \frac{e^{\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}-\frac{k_{1,0}+k_{2,0}}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\chi(k_{1,0}>0)\chi(k_{2,0}>0) \left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(k_{1,0}+k_{2,0}>-\pi T) - e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(k_{1,0}+k_{2,0}>\pi T)\right] . \quad \text{(VII.108)}$$

We remark that the conditions $\chi(k_{1,0}+k_{2,0} > \pm\pi T)$ are superfluous, and that there is no need to introduce the variable s. We have :

$$\sum_{n=0}^{+\infty} e^{(2n+1)\left(\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}-\frac{1}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)Tx_+} \sum_{p=0}^{+\infty} e^{(2p+1)\left(\frac{1}{\cos(\frac{\pi}{2}k_{2,-})}-\frac{1}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)Tx_+}$$

$$\left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}} - e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\right] =$$

$$\frac{e^{-\left(\frac{1}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}-\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}-\frac{1}{\cos(\frac{\pi}{2}k_{2,-})}\right)Tx_+}\left[e^{\frac{-2Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}} - 1\right]}{\left[1-e^{-2\left(\frac{1}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}-\frac{1}{\cos(\frac{\pi}{2}k_{1,-})}\right)Tx_+}\right]\left[1-e^{-2\left(\frac{1}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}-\frac{1}{\cos(\frac{\pi}{2}k_{2,-})}\right)Tx_+}\right]} \; . \quad \text{(VII.109)}$$

This quantity is negative and we conclude that

$$iA_1^{(-,-,+)}(T) \leq 0 \; . \quad \text{(VII.110)}$$

VII.2.7 The $(-,-,-)$ case

We finally discuss the last case :

$$A_1^{(-,-,-)}(T) = -32i \sum_{x_+\in\frac{\pi}{2}\mathbb{N}^*} \int dk_{1,0}\, dk_{2,0} \iint_{\mathscr{T}_{(-,-,-)}} dk_{1,-}\, dk_{2,-}$$

$$x_+^2 \frac{e^{\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}-\frac{k_{1,0}+k_{2,0}}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\,\chi(k_{1,0}>0)\,\chi(k_{2,0}>0)$$

$$\left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(k_{1,0}+k_{2,0}<-\pi T) - e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi(k_{1,0}+k_{2,0}<\pi T)\right] \; . \quad \text{(VII.111)}$$

But it is clear that the conditions $k_{1,0} > 0$, $k_{2,0} > 0$ and $k_{1,0}+k_{2,0} < \pm\pi T$ are incompatible (as in the $(+,+,+)$ case), hence

$$A_1^{(-,-,-)}(T) = 0 \; . \quad \text{(VII.112)}$$

Lemma VII.1 *There exists a constant $K > 0$ such that :*

$$\left|A_1^{(+,+,-)}(T)+A_1^{(+,-,+)}(T)+A_1^{(+,-,-)}(T)+A_1^{(-,-,+)}(T)\right| > \frac{K}{T} \; . \quad \text{(VII.113)}$$

Proof : As each one of the quantities are purely imaginary, with non-negative imaginary part, it is sufficient to prove the inequality $|A_1^{(+,+,-)}(T)| > \frac{K_1}{T}$ for some constant K_1. We have :

$$|A_1^{(+,+,-)}(T)| = 32 \sum_{x_+ \in \frac{\pi}{2}\mathbb{N}^*} \int\int_{\mathscr{T}^{(+,+,-)}} dk_{1,-} dk_{2,-}$$
$$x_+^2 \frac{e^{-\left(\frac{1}{\cos\frac{\pi}{2}k_{1,-}} + \frac{1}{\cos\frac{\pi}{2}k_{2,-}} - \frac{1}{\cos\frac{\pi}{2}(k_{1,-}+k_{2,-})}\right)Tx_+} \left[1 - e^{\frac{2Tx_+}{\cos\frac{\pi}{2}(k_{1,-}+k_{2,-})}}\right]}{\left[1 - e^{-2\left(\frac{1}{\cos\frac{\pi}{2}k_{1,-}} - \frac{1}{\cos\frac{\pi}{2}(k_{1,-}+k_{2,-})}\right)Tx_+}\right]\left[1 - e^{-2\left(\frac{1}{\cos\frac{\pi}{2}k_{2,-}} - \frac{1}{\cos\frac{\pi}{2}(k_{1,-}+k_{2,-})}\right)Tx_+}\right]} . \quad \text{(VII.114)}$$

As $\left[1 - e^{-2\left(\frac{1}{\cos\frac{\pi}{2}k_{1,-}} - \frac{1}{\cos\frac{\pi}{2}(k_{1,-}+k_{2,-})}\right)Tx_+}\right] \leq 1$ and $\left[1 - e^{-2\left(\frac{1}{\cos\frac{\pi}{2}k_{2,-}} - \frac{1}{\cos\frac{\pi}{2}(k_{1,-}+k_{2,-})}\right)Tx_+}\right] \leq 1$, we get :

$$|A_1^{(+,+,-)}(T)| = 32 \sum_{x_+ \in \frac{\pi}{2}\mathbb{N}^*} \int\int_{\mathscr{T}^{(+,+,-)}} dk_{1,-} dk_{2,-}$$
$$x_+^2 e^{-\left(\frac{1}{\cos\frac{\pi}{2}k_{1,-}} + \frac{1}{\cos\frac{\pi}{2}k_{2,-}} - \frac{1}{\cos\frac{\pi}{2}(k_{1,-}+k_{2,-})}\right)Tx_+} \left[1 - e^{\frac{2Tx_+}{\cos\frac{\pi}{2}(k_{1,-}+k_{2,-})}}\right] . \quad \text{(VII.115)}$$

As we are seeking a lower bound, we can restrict the integration over the open domain $\mathscr{T}^{(+,+,-)}$ to a compact $\mathscr{T}_\varepsilon^{(+,+,-)} \subset \mathscr{T}^{(+,+,-)}$, where ε is a strictly positive constant (for example $\varepsilon = \frac{1}{10}$), in which we have $\cos\left(\frac{\pi}{2}k_{1,-}\right) \geq \varepsilon$, $\cos\left(\frac{\pi}{2}k_{2,-}\right) \geq \varepsilon$ and $\left|\cos\left(\frac{\pi}{2}(k_{1,-}+k_{2,-})\right)\right| \geq \varepsilon$. For $(k_{1,-}, k_{2,-}) \in \mathscr{T}_\varepsilon^{(+,+,-)}$, we have :

$$0 < x_+^2 . e^{-\left(\frac{1}{\cos\frac{\pi}{2}k_{1,-}} + \frac{1}{\cos\frac{\pi}{2}k_{2,-}} - \frac{1}{\cos\frac{\pi}{2}(k_{1,-}+k_{2,-})}\right)Tx_+} \left[1 - e^{\frac{2Tx_+}{\cos\frac{\pi}{2}(k_{1,-}+k_{2,-})}}\right] \leq e^{-3Tx_+} . \quad \text{(VII.116)}$$

By Lebesgue domination theorem, we can invert $\sum_{x_+}$ and $\int\int_{\mathscr{T}_\varepsilon^{(+,+,-)}} dk_{1,-}\ dk_{2,-}$ and write :

$$|A_1^{(+,+,-)}(T)| \geq 32 \int_{\mathscr{T}_\varepsilon^{(+,+,-)}} dk_{1,-}\ dk_{2,-} \sum_{x_+ \in \frac{\pi}{2}\mathbb{N}^*} x_+^2 . e^{-\frac{3}{\varepsilon}Tx_+} \left[1 - e^{-2Tx_+}\right] , \quad \text{(VII.117)}$$

or :

$$|A_1^{(+,+,-)}(T)| \geq 8\pi^2 \sum_{n=0}^{+\infty} n^2 e^{-\frac{3\pi}{2\varepsilon}Tn} \left(1 - e^{-\pi T n}\right) . \quad \text{(VII.118)}$$

Now we use the formula :

$$\sum_{n=0}^{+\infty} n^2 e^{-an} = \frac{e^{-a} + e^{-2a}}{(1-e^{-a})^3} \text{ (for } a > 0) \quad \text{(VII.119)}$$

to write :

$$|A_1^{(+,+,-)}(T)| \geq 8\pi^2 \left(\frac{e^{-\frac{3\pi}{2\varepsilon}T} + e^{-\frac{3\pi}{\varepsilon}T}}{\left(1-e^{-\frac{3\pi}{2\varepsilon}T}\right)^3} - \frac{e^{-\left(\frac{3\pi}{2\varepsilon}+1\right)T} + e^{-\left(\frac{3\pi}{\varepsilon}+2\right)T}}{\left(1-e^{-\left(\frac{3\pi}{\varepsilon}+2\right)T}\right)^3} \right) \quad \text{(VII.120)}$$

$$\geq 8\pi^2 \left(\frac{e^{-\frac{3\pi}{2\varepsilon}T} + e^{-\frac{3\pi}{\varepsilon}T} - e^{-\left(\frac{3\pi}{2\varepsilon}+1\right)T} - e^{-\left(\frac{3\pi}{\varepsilon}+2\right)T}}{\left(1-e^{-\frac{3\pi}{2\varepsilon}T}\right)^3} \right) . \quad \text{(VII.121)}$$

Using the inequality $\frac{1}{1-e^{-\frac{3\pi}{2\varepsilon}T}} \geq \frac{2\varepsilon}{3\pi T}$ and assuming that $T < 1$, we obtain the desired result :

$$|A_1^{(+,+,-)}(T)| \geq 8\pi^2 \frac{(2\varepsilon)^3}{(3\pi)^3} \frac{e^{-\frac{3\pi}{2\varepsilon}}(1-e^{-1}) + e^{-\frac{3\pi}{\varepsilon}}(1-e^{-2})}{T^3} . \quad \text{(VII.122)}$$

Therefore the lemma is proven.

VIII Study of the other configurations

We now are going to treat the other configuration, corresponding to :

$$\left\{ \begin{array}{lll} k_{1,+} & \approx & -1 \\ k_{2,+} & \approx & 1 \\ k_{3,+} & \approx & -1 \end{array} \right. \text{ and } \left\{ \begin{array}{lll} k_{1,+} & \approx & -1 \\ k_{2,+} & \approx & -1 \\ k_{3,+} & \approx & 1 \end{array} \right. \quad \text{(VIII.123)}$$

which are equal and form the term called $2\partial_+^2 A_{G,2}(\pi T,1,0)$. Let us concentrate on the first case. We have to consider the propagator :

$$\int_{-\infty}^{+\infty} dk_{1,+} \frac{e^{ik_{1,+}x_+}}{ik_{1,0} - \pi k_{1,+}\cos(\frac{\pi}{2}k_{1,-})} = \frac{-1}{\pi\cos(\frac{\pi}{2}k_{1,-})} \int_{-\infty}^{+\infty} dk_{1,+} \frac{e^{ik_{1,+}x_+}}{k_{1,+} - \frac{ik_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}} . \quad \text{(VIII.124)}$$

The pole of the integrand is $\frac{ik_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}$ and the corresponding residue writes $e^{-\frac{k_{1,0}x_+}{\pi\cos(\frac{\pi}{2}k_{1,-})}}$. Therefore we have :

$$\int_{-\infty}^{+\infty} dk_{1,+} \frac{e^{ik_{1,+}x_+}}{ik_{1,0} - \pi k_{1,+}\cos(\frac{\pi}{2}k_{1,-})} = \frac{-2i}{\cos(\frac{\pi}{2}k_{1,-})} e^{-\frac{k_{1,0}x_+}{\pi\cos(\frac{\pi}{2}k_{1,-})}}$$
$$\left[\chi(x_+>0)\chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})} > 0\right) - \chi(x_+<0)\chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})} < 0\right)\right] . \quad \text{(VIII.125)}$$

Now, let us consider the integration over $k_{2,+}$. We have :

$$\int_{-\infty}^{+\infty} dk_{2,+} \frac{e^{ik_{2,+}x_+}}{-ik_{2,0} + \pi k_{2,+}\cos(\frac{\pi}{2}k_{2,-})} = \frac{1}{\pi\cos(\frac{\pi}{2}k_{2,-})} \int_{-\infty}^{+\infty} dk_{2,+} \frac{e^{ik_{2,+}x_+}}{k_{2,+} - \frac{ik_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}} . \quad \text{(VIII.126)}$$

In fact, the only change with the previous case is a global change of sign. We can immediately write :

$$\int_{-\infty}^{+\infty} dk_{2,+} \frac{e^{ik_{2,+}x_+}}{-ik_{2,0} + \pi k_{2,+} \cos(\frac{\pi}{2}k_{2,-})} = \frac{2i}{\cos(\frac{\pi}{2}k_{2,-})} e^{-\frac{k_{2,0}x_+}{\pi\cos(\frac{\pi}{2}k_{2,-})}}$$
$$\left[\chi(x_+ > 0)\chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})} > 0\right) - \chi(x_+ < 0)\chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})} < 0\right)\right] . \quad \text{(VIII.127)}$$

For the integration over $k_{3,+}$, we have $\cos(\frac{\pi}{2}k_{2,+}) \approx \frac{\pi}{2}(k_{2,+}+1)$ and we consider :

$$\int_{-\infty}^{+\infty} dk_{3,+} \frac{e^{ik_{3,+}x_+}}{-ik_{3,0} - \pi k_{3,+} \cos(\frac{\pi}{2}k_{3,-})} = \frac{-1}{\pi\cos(\frac{\pi}{2}k_{3,-})} \int_{-infty}^{+\infty} \frac{e^{ik_{3,+}x_+}}{k_{3,+} + \frac{ik_{3,0}}{\pi\cos(\frac{\pi}{2}k_{3,+})}} . \quad \text{(VIII.128)}$$

In this case, the pole is $-\frac{ik_{3,0}}{\pi\cos(\frac{\pi}{2}k_{3,-})}$ and the residue $e^{\frac{k_{3,0}x_+}{\pi\cos(\frac{\pi}{2}k_{3,-})}}$. Therefore the above integral writes :

$$\int_{-\infty}^{+\infty} dk_{3,+} \frac{e^{ik_{3,+}x_+}}{-ik_{3,0} - \pi k_{3,+} \cos(\frac{\pi}{2}k_{3,-})} = \frac{-2i}{\cos(\frac{\pi}{2}k_{3,-})} e^{\frac{k_{3,0}x_+}{\pi\cos(\frac{\pi}{2}k_{3,-})}}$$
$$\left[\chi(x_+ > 0)\chi\left(\frac{k_{3,0}}{\pi\cos(\frac{\pi}{2}k_{3,-})} < 0\right) - \chi(x_+ < 0)\chi\left(\frac{k_{3,0}}{\pi\cos(\frac{\pi}{2}k_{3,-})} > 0\right)\right] . \quad \text{(VIII.129)}$$

Hence we obtain :

$$\partial_+^2 \tilde{A}_{G,2}(\pi T, 1, 0) = -8i \int dx \int dk_{1,0}\, dk_{1,-}\, dk_{2,0}\, dk_{2,-}\, dk_{3,0}\, dk_{3,-}$$
$$x_+^2 \frac{e^{\left(-\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})} - \frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})} + \frac{k_{3,0}}{\pi\cos(\frac{\pi}{2}k_{3,-})}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}k_{3,-})} e^{i(k_{1,0}+k_{2,0}+k_{3,0}+\pi T)t}\, e^{i(k_{1,-}+k_{2,-}+k_{3,-})x_-}$$
$$\left[\chi(x_+ > 0)\,\chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})} > 0\right)\chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})} > 0\right)\chi\left(\frac{k_{3,0}}{\pi\cos(\frac{\pi}{2}k_{3,-})} < 0\right)\right.$$
$$\left. - \chi(x_+ < 0)\,\chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})} < 0\right)\chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})} < 0\right)\chi\left(\frac{k_{3,0}}{\pi\cos(\frac{\pi}{2}k_{3,-})} > 0\right)\right] . \quad \text{(VIII.130)}$$

Then we integrate over t and perform the sum over $k_{3,0}$:

$$\partial_+^2 \tilde{A}_{G,2}(\pi T,1,0) = -8i \int dx_+\, dx_- \int dk_{1,0}\, dk_{1,-}\, dk_{2,0}\, dk_{2,-}\, dk_{3,-}$$
$$x_+^2 \frac{e^{\left(-\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}-\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}-\frac{k_{1,0}+k_{2,0}+\pi T}{\pi\cos(\frac{\pi}{2}k_{3,-})}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}k_{3,-})} e^{i(k_{1,-}+k_{2,-}+k_{3,-})x_-}$$
$$\left[\chi(x_+>0)\,\chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}>0\right)\chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}>0\right)\chi\left(\frac{k_{3,0}}{\pi\cos(\frac{\pi}{2}k_{3,-})}<0\right)\right.$$
$$\left.-\,\chi(x_+<0)\,\chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}<0\right)\chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}<0\right)\chi\left(\frac{k_{3,0}}{\pi\cos(\frac{\pi}{2}k_{3,-})}>0\right)\right]\,. \tag{VIII.131}$$

Thanks to the change of variables $x'_+ = -x_+$, $k'_{1,0} = -k_{1,0}$, $k'_{2,0} = -k_{2,0}$, we get :

$$\partial_+^2 \tilde{A}_{G,2}(\pi T,1,0) = -8i \int dx_+\, dx_- \int dk_{1,0}\, dk_{1,-}\, dk_{2,0}\, dk_{2,-}\, dk_{3,-}$$
$$x_+^2 \frac{e^{-\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}+\frac{k_{1,0}+k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{3,-})}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}k_{3,-})} e^{i(k_{1,-}+k_{2,-}+k_{3,-})x_-}$$
$$\chi(x_+>0)\,\chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}>0\right)\chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}>0\right)$$
$$\left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}k_{3,-})}}\chi\left(\frac{k_{1,0}+k_{2,0}+\pi T}{\pi\cos(\frac{\pi}{2}k_{3,-})}>0\right)-e^{\frac{Tx_+}{\cos(\frac{\pi}{2}k_{3,-})}}\chi\left(\frac{k_{1,0}+k_{2,0}-\pi T}{\pi\cos(\frac{\pi}{2}k_{3,-})}>0\right)\right]\,. \tag{VIII.132}$$

Then we perform the sum over x_- as previously and integrate over $k_{3,-}$. There is a small contribution with a compensating factor $[\chi(x_+\ even)-\chi(x_+\ odd)]$ that can be bounded as in Section VI, and we have again to study the dominant contribution :

$$A_2(T) = -8i \sum_{x_+\in\frac{\pi}{2}\mathbb{N}^*} \int dk_{1,0}\, dk_{2,0}\, dk_{1,-}\, dk_{2,-} x_+^2 \frac{e^{-\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}+\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}+\frac{k_{1,0}+k_{2,0}}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}\right)x_+}}{\cos(\frac{\pi}{2}k_{1,-})\cos(\frac{\pi}{2}k_{2,-})\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}$$
$$\chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k_{1,-})}>0\right)\chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k_{2,-})}>0\right)\left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi\left(\frac{k_{1,0}+k_{2,0}+\pi T}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}>0\right)\right.$$
$$\left.-\,e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}}\chi\left(\frac{k_{1,0}+k_{2,0}-\pi T}{\pi\cos(\frac{\pi}{2}(k_{1,-}+k_{2,-}))}>0\right)\right]\,. \tag{VIII.133}$$

Fortunately, we do not have to carry again a discussion about the signs of the three cosines. In fact, we can remark that $A_1(T) = A_2(T)$. To see that, let us perform the following change of variables in $A_1(T)$:

$$\left\{\begin{array}{rcl} k_{1,-} &=& k'_{1,-}+2 \\ k_{2,-} &=& k'_{2,-}+2 \end{array}\right. , \tag{VIII.134}$$

to obtain :

$$
A_2(T) = -8i \sum_{x_+ \in \frac{\pi}{2}\mathbb{N}^*} \int dk_{1,0}\, dk_{2,0} \iint_{\mathscr{T}'} dk'_{1,-} dk'_{2,-} x_+^2 \frac{e^{\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k'_{1,-})} + \frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k'_{2,-})} - \frac{k_{1,0}+k_{2,0}}{\pi\cos(\frac{\pi}{2}(k'_{1,-}+k'_{2,-}))}\right)x_+}}{\cos(\frac{\pi}{2}k'_{1,-})\cos(\frac{\pi}{2}k'_{2,-})\cos(\frac{\pi}{2}(k'_{1,-}+k'_{2,-}))}
$$

$$
\chi\left(\frac{k_{1,0}}{\pi\cos(\frac{\pi}{2}k'_{1,-})} < 0\right) \chi\left(\frac{k_{2,0}}{\pi\cos(\frac{\pi}{2}k'_{2,-})} < 0\right) \left[e^{\frac{-Tx_+}{\cos(\frac{\pi}{2}(k'_{1,-}+k'_{2,-}))}} \chi\left(\frac{k_{1,0}+k_{2,0}+\pi T}{\pi\cos(\frac{\pi}{2}(k'_{1,-}+k'_{2,-}))} > 0\right)\right.
$$

$$
\left. - e^{\frac{Tx_+}{\cos(\frac{\pi}{2}(k'_{1,-}+k'_{2,-}))}} \chi\left(\frac{k_{1,0}+k_{2,0}-\pi T}{\pi\cos(\frac{\pi}{2}(k'_{1,-}+k'_{2,-}))} > 0\right)\right] , \quad \text{(VIII.135)}
$$

where $\mathscr{T}'$ is the triangle $\mathscr{T}$ translated by the vector $(-2,-2)$. Using the invariance under central symmetry and translations by vectors of the form $(4n_+, 4n_-)$, $(n_+, n_-) \in \mathbb{Z}^2$, we conclude that $\mathscr{T}'$ may be replaced by $\mathscr{T}$.

Hence we have proved that $A_1(T) = A_2(T)$. This concludes the proof of Theorem IV.1 hence of Theorem II.1.

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